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The Normalized CES Production Function: Theory and Empirics

By RAINER KLUMP*, PETER MCADAM** and ALPO WILLMAN***

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Abstract

The elasticity of substitution between capital and labor and, in turn, the direction of technical change is critical parameters in many fields of economics. Until recently, though, the application of production functions with specifically non-unitary substitution elasticities (i.e., non-Cobb–Douglas) was hampered by empirical and theoretical uncertainties. As recently revealed, “normalization” of production-technology systems holds out the promise of resolving many of those uncertainties. We survey and assess the intrinsic links between production (as conceptualized in a production function), factor substitution (as made most explicit in Constant Elasticity of Substitution functions) and normalization (defined by the fixing of baseline values for relevant variables). First, we recall how the normalized Constant Elasticity of Substitution function came into existence and what normalization implies for its formal properties. Then we deal with the key role of normalization in recent advances in the theory of business cycles and of economic growth. Next, we discuss the benefits normalization brings for empirical estimation and empirical growth research. Finally, we identify promising areas of future research.

Keywords: Constant Elasticity of Substitution Production Function, Estimation, Factor-Augmenting Technical Change, Growth Theory, Identification, Normalization

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1. Introduction

Substituting scarce factors of production by relatively more abundant ones is a key element of economic efficiency and a driving force of economic growth. A measure of that force is the elasticity of substitution between capital and labor which is the central parameter in production functions, and in particular Constant Elasticity of Substitution (CES) ones. CES production functions allow the elasticity of substitution to be any positive number; in contrast, the more well-known Cobb–Douglas variant imposes that elasticity to be unity.

Until recently, the application of production functions with non-unitary substitution elasticities was hampered by empirical and theoretical uncertainties. As has recently been revealed, ‘normalization’ of production functions and production-technology systems holds out the promise of resolving many of those uncertainties and allowing considerations such as the role of the substitution elasticity and biased technical change to play a deeper role in growth and business-cycle analysis.

Normalization essentially implies representing production relations in consistent indexed number form. Without normalization, it can be shown that the production function parameters have no economic interpretation since they are dependent on the normalization point and the elasticity of substitution itself. This feature significantly undermines estimation and comparative-static exercises, among other things.

Notwithstanding, this leaves open the issue of how we set and interpret normalization points. All CES production functions, unless explicitly normalized, are at least implicitly normalized in the point where input values equal one. However, that implicit normalization is empirically counter-factual and, from a theory standpoint, unattractive. Accordingly, normalization points tend to be pinned down by some prevailing economic theory or empirical counterpart – typically some steady-state or initial condition, or in some neighborhood of particular interest to the researcher. Beyond that, normalization (explicit normalization) also implies internal consistency between other aspects of the data or model. As we shall see, if this internal consistency condition is violated – as typically has been the case in theoretic analysis based on the non-normalized (or ‘trivially’ normalized) CES function – then analysis concerning the effects of alternative elasticity of substitution values on economic development is flawed.

Let us first, though, place the importance of the topic in perspective. Due to the central role of the substitution elasticity in many areas of dynamic macroeconomics, the concept of CES production functions has recently experienced a major revival. The link between economic growth and the size of the substitution elasticity has long been known. As already demonstrated by Solow (1956) in the neoclassical growth model, assuming an aggregate CES production function with an elasticity above unity is the easiest way to generate perpetual growth. Since scarce labor can be completely substituted by capital, the marginal product of capital remains bounded away from zero in the long run. Nonetheless, as we argue below, the case for an above-unity elasticity appears empirically weak and theoretically anomalous.¹

¹ The critical threshold level for the substitution elasticity (to generate such perpetual growth) can be shown to be

It has been shown that integration into world markets is also a feasible way for a country to increase the effective substitution between factors of production and thus pave the way for sustained growth (Ventura 1997; Klump 2001; Saam 2008). On the other hand, it can be shown in several variants of the standard neoclassical (exogenous) growth model that introducing an aggregate CES production function with an elasticity of substitution below unity can generate multiple growth equilibria, development traps and indeterminacy (Azariadis 1996; Klump 2002; Kaas and von Thadden 2003; Guo and Lansing 2009).

Public finance and labor economics are other fields where the elasticity of substitution has been rediscovered as a crucial parameter for understanding the impact of policy changes. This relates to the importance of factor substitution possibilities for the demand for each input factor. As pointed out by Chirinko (2002), the lower the elasticity of substitution, the smaller the response of business investment to variations in interest rates caused by monetary or fiscal policy.² In addition, the welfare effects of tax policy changes, specifically, appear highly sensitive to the values of the substitution elasticity. Rowthorn (1999) also stresses its importance in macroeconomic analysis of the labor market and, in particular, how incentives for investment exercise a significant effect on unemployment when the elasticity of substitution departs from unity.

Indeed, there is now mounting empirical evidence that aggregate production is better characterized by a non-unitary (and in particular below unitary) elasticity of substitution (see e.g., Chirinko *et al.* 1999; Klump *et al.* 2007; Leon-Ledesma *et al.* 2010a). Chirinko (2008)'s recent survey suggests that most evidence favors elasticities ranges of 0.4–0.6 for the United States. Moreover, Jones (2003, 2005)³ argued that capital shares exhibit such protracted swings and trends in many countries as to be inconsistent with Cobb–Douglas or CES with Harrod-neutral technical progress (see also Blanchard 1997; McAdam and Willman 2013). Such variability would also suggest the presence of biases in technical change.

The coexistence of capital and labor-augmenting technical change has different implications for the possibility of balanced or unbalanced growth. A balanced growth path (BGP) – the dominant assumption in the theoretical growth literature – suggests that variables such as output, consumption, etc. tend to a common growth rate, whilst key underlying ratios (e.g., factor income shares, capital–output ratio) are constant (Kaldor 1961). Neoclassical growth theory suggests that, for an economy to possess a steady state with positive growth and constant factor income shares, the elasticity of substitution must be unitary (i.e., Cobb–Douglas) or technical change must be Harrod-neutral.

As Acemoglu (2009) (ch. 15) comments, however, there is little reason to assume technical change is necessarily labor-augmenting.⁴ In models of ‘biased’ technical

increasing in the growth of labor force and decreasing in the saving rate (see La Grandville 1989b).

² This may be one reason why estimated investment equations struggle to identify interest-rate channels.

³ Jones' work essentially builds on Houthakker (1955)'s idea that production combinations reflect the (Pareto) distribution of innovation activities, Jones proposes a ‘nested’ production function. Given such parametric innovation activities, this will exhibit a (far) less than unitary substitution elasticity over business-cycle frequencies but asymptote to Cobb–Douglas.

⁴ Moreover, the point that a BGP cannot coexist with capital augmentation is becoming increasingly questioned in

change (e.g., Kennedy 1964; Samuelson 1965; Acemoglu 2003; Sato 2006), scarcity, reflected by relative factor prices, generates incentives to invest in factor-saving innovations. In other words, firms reduce the need for scarce factors and increase the use of abundant ones. Acemoglu (2003) further suggested that while technical progress is necessarily labor-augmenting along the BGP, it may become capital-biased in transition. Interestingly, given a below-unitary substitution elasticity this pattern promotes the stability of income shares while allowing them to fluctuate in the medium run.

However, when analytically investigating the significance of non-unitary factor substitution and non-neutral technical change in dynamic macroeconomic models, one faces the issue of ‘normalization’, even though the issue is still not widely known. The (re)discovery of the CES production function in normalized form in fact paved the way for the new and fruitful, theoretical and empirical research on the aggregate elasticity of substitution which has been witnessed over recent years.

In La Grandville (1989b) and Klump and de La Grandville (2000) the concept of normalization was introduced in order to prove that the aggregate elasticity of substitution between labor and capital can be regarded as an important and meaningful determinant of growth in the neoclassical growth model. In the meantime this approach has been successfully applied in a series of theoretical papers (Klump 2001; Papageorgiou and Saam 2008; Klump and Irmen 2009; Xue and Yip 2013; Guo and Lansing 2009; Wong and Yip 2010) to a wide variety of topics.

A particular striking example of how neglecting normalization can significantly bias results and how explicit normalization can help to overcome those biases is presented in Klump and Saam (2008). The effect of a higher elasticity of substitution on the speed of convergence in a standard Ramsey type growth model is shown to double if an implicitly normalized CES function is replaced by a reasonably normalized one.

Further, as Klump *et al.* (2007, 2008) demonstrated, normalization also has been a significant development for empirical research on the parameters of aggregate CES production functions,⁵ in particular when coupled with the system estimation approach. Empirical research has long been hampered by the difficulties in identifying at the same time an aggregate elasticity of substitution as well as growth rates of factor-augmenting technical change from the data. Following Diamond *et al.* (1978), the received wisdom suggests that their joint identification was infeasible. Accordingly, for more than a quarter of a century following Berndt (1976), common opinion held that the US economy was broadly characterized by aggregate Cobb–Douglas technology, leading, in turn, to its default incorporation in economic models (and, accordingly, the neglect of possible biases in technical progress in empirical work).⁶

the literature (see Growiec 2008; La Grandville 2012; Leon-Ledesma and Satchi 2010).

⁵ It should be noted that the advantages of re-scaling input data to ease the computational burden of highly nonlinear regressions has been the subject of some study (e.g., ten Cate 1992). And some of this work was in fact framed in terms of production-function analysis (De Jong 1967; De Jong and Kumar (1972)). See also Cantore and Levine (2011).

⁶ It should be borne in mind, however, that Berndt’s result concerned only the US manufacturing sector.

Translating normalization into empirical production-technology estimations allows the presetting of the value of the distribution parameter (or, if estimated, facilitates the setting of reasonable initial conditions); it provides a clear correspondence between theoretical and empirical production function parameters and allows us *ex post* validation of estimated parameters. In a series of papers, Leon-Ledesma *et al.* (2010a, 2010b) showed the empirical advantages in estimating and identifying production technology systems when normalized. Further, McAdam and Willman (2011b) showed that normalized factor-augmenting CES estimation, in the context of estimating ‘New Keynesian’ Phillips curves, helped better identify the volatility in the driving variable (real marginal costs) that most previous researchers had not detected.

Here, we analyze the intrinsic links between production (as conceptualized in a production function), factor substitution (as made most explicit in CES production functions) and normalization. The paper is organized as follows. In Section 2 we recall how the CES function came into existence and what this implies for its formal properties. Sections 3 and 4 will deal with the role of normalization in recent advances in the theory of business cycles and economic growth. Section 5 will discuss the merits normalization brings for empirical growth research. The last section concludes and identifies promising area of future research.

2. The Normalized CES Production Function and Variants

It is common knowledge that the first rigid derivation of the CES production function appeared in the famous Arrow *et al.* (1961) paper (hereafter *ACMS*).⁷ However, there were important forerunners, in particular the explicit mentioning of a CES-type production technology (with an elasticity of substitution equal to 2) in the Solow (1956) paper (done, Solow wrote, to add a ‘bit of variety’) on the neoclassical growth model. There was also the hint to a possible CES function in its Swan (1956) counterpart (on the Swan story see Dimond and Spencer 2008).⁸ Shortly before, though, Dickinson (1954) (p. 169, fn 1) had already made use of a CES production technology in order to model ‘a more general kind of national-income function, in which the factor shares are variable’ compared to the Cobb–Douglas form. It has even been conjectured that the famous and mysterious tombstone formula of von Thunen dealing with ‘just wages’ can be given a meaningful economic interpretation if it is regarded as derived from an implicit CES production function with an elasticity of substitution equal to 2 (see Jensen 2011).

In this section we want to demonstrate that the formal construction of a CES production function is intrinsically linked to normalization, and how it is linked. The function may be defined as follows:

$$Y_t = F(K_t, N_t) = C \left[\pi K_t^{\frac{\sigma-1}{\sigma}} + (1 - \pi) N_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (1)$$

⁷ It is still not widely known that the famous *ACMS* paper was in fact the merging of two separate submissions to the *Review of Economics and Statistics* following a paper from Arrow and Solow, and another from Chenery and Minhas.

⁸ In the inaugural ANU Trevor Swan Distinguished Lecture, Peter L. Swan (Swan 2006) writes, ‘While Trevor was at MIT he pointed out that a production function Solow was utilizing had the constant elasticity of substitution, CES, property. In this way, the CES function was officially born. Solow and his coauthors publicly thanked Trevor for this insight (see Arrow *et al.*, 1961)’.

where distribution parameter $\pi \in (0, 1)$ reflects capital intensity in production; C is an efficiency parameter, and σ is the elasticity of substitution between capital, K , and labor, N . Like all standard CES functions, equation (1) nests a Cobb–Douglas function when $\sigma = 1$; a Leontief function with fixed factor proportions when $\sigma = 0$ and a linear production function with perfect factor substitution when $\sigma \rightarrow \infty$.

The construction of such an aggregate production technology with a CES property starts from the formal definition of the elasticity of substitution which had been introduced independently by Hicks (1932) and Robinson (1933) (on the differences between the two approaches to the concept, see Hicks 1970). It is there defined (in the case of two factors of production, capital and labor) as the elasticity of K/N with respect to the marginal rate of substitution between K and N (the percentage change in factor proportions due to a change in the marginal rate of technical substitution) along an isoquant:⁹

$$\sigma \in [0, \infty) = \frac{d(K/N)/(K/N)}{d(F_N/F_K)/(F_N/F_K)} = \frac{d \log(K/N)}{d \log(F_N/F_K)} \quad (2)$$

As Hicks notes this concept of elasticity can be equally expressed in terms of the second derivative of the production function, but only under the assumption of constant returns to scale (due to Euler's theorem).

Since under this assumption the marginal factor productivities would also equal factor prices and the marginal rate of substitution would be identical with the wage/capital rental ratio, the elasticity of substitution can also be expressed as the elasticity of income per person y with respect to the marginal product of labor in efficiency terms (or the real wage rate, w), that is, Allen's theorem (Allen 1938). Given that income per person is a linear homogeneous function $y = f(k)$ of the capital intensity $k = K/N$, the elasticity of substitution can *also* be defined as:

$$\sigma = \frac{dy}{dw} \cdot \frac{w}{y} = - \frac{f'(k)[f(k) - kf'(k)]}{kf''(k)f(k)} \quad (3)$$

Although it is rarely stated explicitly, the elasticity of substitution is implicitly always defined as a point elasticity. This means that it is related to one particular baseline point on one particular isoquant (see Figures 1 and 2). From there a whole system of non-intersecting isoquants is defined which all together create the CES production function. Even if it is true that a given and constant elasticity of substitution would not change along a given isoquant or within a given system of isoquants, it is also evident that changes in the elasticity of substitution would of course alter the system of isoquants.

Following such a change in the elasticity of substitution, the old and the new isoquant are not intersecting at the baseline point but are tangents, if the production function is normalized (by the values of the baseline points). And they should not intersect because given the definition of the elasticity of substitution (i.e., the percentage change in factor proportions due to a change in the marginal rate of technical substitution) at this particular point (as in all other points which are characterized by the same factor proportion) the old and the new CES function should

⁹ Alternatively, the substitution elasticity is sometimes expressed in terms of the parameter of factor substitution, $\rho \in [-1, \infty)$, where $\rho = \frac{1-\sigma}{\sigma}$.

still be characterized by the same factor proportion and the same marginal rate of technical substitution.

Just as there are two possible definitions of σ following (3) – from $\frac{dy}{dw} \cdot \frac{w}{y}$ and then from $-\frac{f'(k)[f(k)-kf'(k)]}{kf''(k)f(k)}$ – thus there are two ways of uncovering the normalized production function. These, we cover in the following two sub-sections.

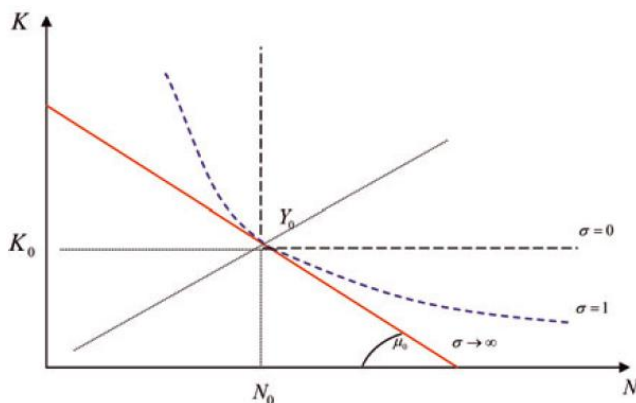


Figure 1. Isoquants of Normalized CES Production Functions.

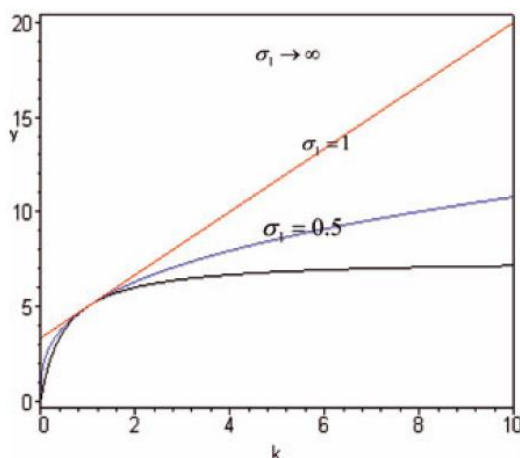


Figure 2. Normalized per-capita CES Production Functions.

2.1 Derivation via the Power Function

Let us start from the definition $\sigma = \frac{d \log(y)}{d \log(w)} = \frac{dy}{dw} \frac{w}{y}$, integration of which gives the power function,

$$y = cw^\sigma \quad (4)$$

where c is some integration constant.¹⁰ Under the assumption of constant returns to

¹⁰ ACMS started from the empirical observation that the relationship between per-capita income and the wage rate might best be described with the help of the power function. Note, $\sigma = 1$ implies a linear relationship between y and w which would, in turn, imply that labor's share of income was constant. However, instead of a linear $y - w$ scatter plot, ACMS found a concave relationship in the US data. The authors then tested a logarithmic and power relationship and concluded in favor of $\sigma < 1$. Integration of power function (4) then leads to a production function with constant elasticity of substitution, consistent with definitions (2) and (3).

scale (or perfectly competitive factor and product markets), and applying the profit-maximizing condition that the real wage equals the marginal product of labor, and with the application of Allen's theorem, we can transform this equation into the form $y = c(y - k \frac{dy}{dk})^\sigma$.

Accordingly, after integration and simplification, this leads us to a production function with the constant elasticity of substitution function (see La Grandville 2009, p. 83ff for further details):

$$y = \left[\beta k^{\frac{\sigma-1}{\sigma}} + \alpha \right]^{\frac{\sigma}{\sigma-1}} \quad (5)$$

and,

$$Y = \left[\beta K^{\frac{\sigma-1}{\sigma}} + \alpha L^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (6)$$

in the extensive form.

It should be noted that (5) and (6) contain the two constants of integration β and $\alpha = c^{\frac{1}{\sigma}}$, where the latter directly depends on σ . Identification of these two constants makes use of baseline values for the power function (4) and for the functional form (5) at the given baseline point in the system of isoquants. In a dynamic setting this baseline point must (as we will see later) also be regarded as holding at a particular point in time, $t = t_0$:

$$y_0 = c w_0^\sigma \quad (7)$$

$$y_0 = \left[\beta k_0^{\frac{\sigma-1}{\sigma}} + \alpha \right]^{\frac{\sigma}{\sigma-1}} \quad (8)$$

Together with (5) this leads to the normalized CES production function,

$$y = y_0 \left[\pi_0 \left(\frac{k}{k_0} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \right]^{\frac{\sigma}{\sigma-1}} \quad (9)$$

and

$$Y = Y_0 \left[\pi_0 \left(\frac{K}{K_0} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \left(\frac{N}{N_0} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (10)$$

in the extensive form. Parameter $\pi_0 = \frac{y_0 - w_0}{y_0} = \frac{r_0 K_0}{Y_0}$ denotes the capital share in total income at the point of normalization.¹¹ As a test of consistent normalization, we see from (10) that for $t = t_0$ we retrieve $Y = Y_0$.

2.2 Derivation via the Homogeneous Production Function

It was shown by Paroush (1964), Yasui (1965) and McElroy (1967) that the rather narrow assumption of Allen's theorem is not essential for the derivation of the CES production function which can start directly from the original Hicks definition (2).

¹¹ Under perfect competition, this distribution parameter is equal to the capital income share but, under *imperfect* competition with non-zero aggregate mark-up, it equals the share of capital income in total factor income.

This definition can be transformed into a second-order differential equation whose solution also implies two constants of integration.

Following Klump and Preissler (2000) we start with the definition of the elasticity of substitution in the case of linear homogeneous production function $Y_t = F(K_t, N_t) = N_t f(k_t)$ where $k_t = K_t/N_t$ is the capital–labor ratio in efficiency units. Likewise $y_t = Y_t/N_t$ represents per-capita production.

The definition of the substitution elasticity, $\sigma = \frac{f'(k)[f(k) - k f'(k)]}{k f'(k) f(k)}$, can then be viewed as a second order differential equation in k having the following general CES production function as its solution (intensive and extensive forms),

$$y_t = a \left[k_t^{\frac{\sigma-1}{\sigma}} + b \right]^{\frac{\sigma}{\sigma-1}} \quad (11)$$

$$Y_t = a \left[K_t^{\frac{\sigma-1}{\sigma}} + b N_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (12)$$

where parameters a and b are two arbitrary constants of integration with the following correspondence with the parameters in equation (1): $C = a(1 + b)^{\frac{\sigma}{\sigma-1}}$ and $\pi = 1/(1 + b)$.

A meaningful identification of these two constants is given by the fact that the substitution elasticity is a point elasticity relying on three baseline values: a given capital intensity $k_0 = K_0/N_0$, a given marginal rate of substitution $[FK/FN]_0 = w_0/r_0$ and a given level of per-capita production $y_0 = Y_0/N_0$. Accordingly, (1) becomes

$$Y_t = Y_0 \left[\pi_0 \left(\frac{K_t}{K_0} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \left(\frac{N_t}{N_0} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (13)$$

where $\pi_0 = r_0 K_0 / (r_0 K_0 + w_0 N_0)$ is the capital income share evaluated at the point of normalization. Rutherford (2003) calls (13) [or (10)] the ‘calibrated form’.

2.3 A Graphical Representation

Normalization as understood by La Grandville (1989b), Klump and de La Grandville (2000) and Klump and Preissler (2000) is again nothing else but identifying these two arbitrary constants in an economically meaningful way. Normalizing means the fixing (in the $K - N$ plane as in Figure 1) of a baseline point (which can be thought of as a point in time at $t = t_0$), characterized by specific values of N , K , Y and the marginal rate of technical substitution μ_0 – in which isoquants of CES functions with different elasticities of substitution but with all other parameters equal – are tangents.

Normalization is helpful to clarify the conceptual relationship between the elasticity of substitution and the curvature of the isoquants of a CES production function (see La Grandville 1989a for a discussion of various misunderstandings on this point). Klump and Irmen (2009) point out that in the point of normalization (and only there), there exists an inverse relationship between the elasticity of substitution and the curvature of isoquant of the normalized CES production function. This relationship also has an interpretation in terms of the degree of complementarity of both input

factors. At the normalization point, a higher elasticity of substitution implies a lower degree of complementarity between the input factors. The link between complementarity between input factors and the elasticity of substitution is also discussed in Acemoglu (2002a) and in Nakamura and Nakamura (2008).

Equivalently (in the $k - y$ plane as in Figure 2) the baseline point can be characterized by specific values of k , y and the marginal productivity of capital (or the real wage rate). If base values for these three variables are selected, this means of course that also a baseline value for the elasticity of production with respect to capital input is fixed which (under perfect competition) is equal to the capital share in total income.

2.4 Normalization as a Means to Uncover Valid CES Representations

Normalization thus creates specific ‘families’ of CES functions whose members all share the same baseline point but are distinguished by the elasticity of substitution (and *only* the elasticity of substitution).

As shown in Klump and Preissler (2000), normalization also helps to distinguish those variants of CES production functions which are functionally identical with the general form (1) from those which are inconsistent with (5) in one way or another.

Consider, first, the ‘standard form’ of the CES production function, as it was introduced by ACMS, restated below:

$$Y = C \left[\pi K^{\frac{\sigma-1}{\sigma}} + (1 - \pi) N^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (14)$$

This variant is clearly identical with (10), albeit (and this is a crucial aspect) with the ‘efficiency parameter’ C and the ‘distribution parameter’ π being defined in the following way (solving for completeness in terms of both σ and the substitution parameter $\rho = \frac{1-\sigma}{\sigma}$):

$$C(\sigma, \cdot) = Y_0 \left[\frac{r_0 K_0^{1+\rho} + w_0 N_0^{1+\rho}}{r_0 K_0 + w_0 N_0} \right]^{-\frac{1}{\rho}} = Y_0 \left[\frac{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}}{r_0 K_0 + w_0 N_0} \right]^{\frac{\sigma}{\sigma-1}} \quad (15)$$

$$\pi(\sigma, \cdot) = \frac{r_0 K_0^{1+\rho}}{r_0 K_0^{1+\rho} + w_0 N_0^{1+\rho}} = \frac{r_0 K_0^{1/\sigma}}{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}} \quad (16)$$

Expressions (15) and (16) reveal that, in the implicitly normalized case, both ‘parameters’ (apart from being dependent on the scale of the normalized variables) change with variations in the elasticity of substitution, unless the particular case of K_0 and N_0 are exactly equal arises, implying $k_0 = 1$.

This makes the implicitly normalized form in general inappropriate for comparative-static exercises in the substitution elasticity. It is the interaction between the normalized efficiency and distribution terms and the elasticity of substitution which guarantees that within one family of CES functions the members are only distinguished by the elasticity of substitution. Given the accounting identity (and abstracting from the absence of an aggregate mark-up),

$$Y_0 = r_0 K_0 + w_0 N_0 \quad (17)$$

it also follows from this analysis that treating C and π in (14) as deep parameters is equivalent to assuming $k_0 = 1$. In the case $\sigma = 0$, we have a perfectly symmetrical Leontief function.

As explained in Klump and Saam (2008) the Leontief case can serve as a benchmark for the choice of the normalization values for k_0 in calibrated growth models. The baseline capital intensity corresponds to the capital intensity that would be efficient if the economy's elasticity of substitution were zero. For $k < k_0$ the economy's relative bottleneck resides in this case in its capacity to make productive use of additional labor, as capital is the relatively scarce factor. For $k > k_0$ the same is true for capital and labor is relatively scarce. Since the latter case is most characteristic for growth model of capitalist economies, calibrations of these model can be based on the assumption $k > k_0$.

In the following sub-sections, we will illustrate how normalization can reveal whether certain production functions used in the literature are legitimate.

2.4.1 David and van de Klundert (1965) Version

Consider the CES variant proposed by David and van de Klundert (1965):

$$Y = \left[(BK)^{\frac{\sigma-1}{\sigma}} + (AN)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (18)$$

This variant is identical with (10) as long as the two 'efficiency levels' are defined in the following way:

$$B = \frac{Y_0}{K_0} \pi_0^{\frac{\sigma}{\sigma-1}} \quad (19)$$

$$A = \frac{Y_0}{N_0} (1 - \pi_0)^{\frac{\sigma}{\sigma-1}} \quad (20)$$

Again, it is obvious that the efficiency levels change directly with the elasticity of substitution.

2.4.2 Ventura (1997) Version

Consider now a CES variant used by Ventura (1997):

$$Y = \left[K^{\frac{\sigma-1}{\sigma}} + (AN)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (21)$$

At first glance (21) could be regarded as a special case of (14) with B being equal to one. With a view on the normalized efficiency level it becomes clear, however, that $B = 1$ is not possible for given baseline values and a changing elasticity of substitution. Given that Ventura (1997) makes use of (21) in order to study the impact of changes in the elasticity of substitution on the speed of convergence, in the light of this inconsistency his results should be regarded with caution. Indeed, as shown in Klump (2001), Ventura's results are unnecessarily restrictive; working with a correctly normalized CES technology leads to much more general results.

2.4.3 Barro and Sala-i-Martin (2004) Version

Next consider the CES production function proposed by Barro and Sala-i-Martin (2004):

$$Y = C \left[\pi (BK)^{\frac{\sigma-1}{\sigma}} + (1-\pi)((1-B)N)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (22)$$

Normalization is helpful in this case in order to show that (22) can be transformed without any problems into (10) and/or (14) so that the terms B and $1 - B$ simply disappear. If for any reason these two terms are considered necessary elements of a standard CES production function, they cannot be chosen independently from the normalized values for C and π , but they remain independent from changes in σ .

2.5 The Normalized CES Function with Technical Progress

So far, we have treated efficiency levels as constant over time. If we now consider factor-augmenting technical progress one has to keep in mind the intrinsic links between rising factor efficiency and the distribution of income. This brings us to one further justification for normalizing CES production function which is closely related to the concept of neutral technical progress and which was first articulated by Kamien and Schwartz (1968).

Normalization implies that there may be considered a reference (or representative) value for the capital income share (and thus for income distribution) at some given point. Technical progress that does not change income distribution over time is called Harrod-neutral (or labor-augmenting) technical progress. There are other types of neutral technical change, however, that would not have this effect.¹² So the whole concept of whether technical progress is neutral with respect to the income distribution relies on the idea that one has to check whether or not a given income distribution at one point in time remains constant. This given income distribution, which is used to evaluate possible distribution effect of technical progress, is exactly the income distribution in the baseline point of normalization at a fixed point in time, $t = t_0$.

2.5.1 Constant growth rates of normalized factor efficiency levels

A CES production function with factor-augmenting technical progress can be written as

$$Y = \left[(E_t^K K)^{\frac{\sigma-1}{\sigma}} + (E_t^N N)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (23)$$

where E_t^K and E_t^N represent the levels of efficiency of both input factors.¹³

Thus, whereas the ACMS specification seems to imply that technological change is always Hicks-neutral, the above specification allows for different growth rates of factor efficiency. To circumvent problems related to Diamond-McFadden's Impossibility theorem (Diamond *et al.* 1978; Diamond and McFadden 1965), we

¹² See the seminal contribution of Sato and Beckmann (1970) for such a classification.

¹³ In the case where there is such technical progress, the question of whether σ is greater than or below unity takes on added importance. Recall, when $\sigma < 1$, factors are 'gross complements' in production and 'gross substitutes' otherwise. Thus, it can be shown that with gross substitutes, substitutability between factors allows both the augmentation and bias of technological change to favor the same factor. For gross complements, however, a capital-augmenting technological change, for instance, increases demand for labor (the complementary input) more than it does capital, and vice versa. By contrast, when $\sigma = 1$ an increase in technology does not produce a bias towards either factor (factor shares will always be constant since any change in factor proportions will be offset by a change in factor prices).

assume a certain functional form for the growth rates of both efficiency levels and define

$$E_t^i = E_0^i e^{\gamma_i(t-t_0)} \quad (24)$$

where γ_i denotes growth in technical progress associated with factor i and t represents time. The combination $\gamma_K = \gamma_N > 0$ denotes Hicks-neutral technical progress; $\gamma_K > 0, \gamma_N = 0$ yields Solow-neutrality; $\gamma_K = 0, \gamma_N > 0$ represents Harrod-neutrality; and $\gamma_K > 0 \neq \gamma_N > 0$ indicates general factor-augmenting technical progress.¹⁴

E_0^i are the fixed points of the two efficiency levels, taken at the common baseline time, $t = t_0$. Again, normalization of the CES function implies that members of the same CES family should all share the same fixed point but differ in their elasticity of substitution values (Figures 1 and 2 showed this graphically). In order to ensure that this property also holds in the presence of growing factor efficiencies, it follows that

$$E_0^N = \frac{Y_0}{N_0} \left(\frac{1}{1 - \pi_0} \right)^{\frac{\sigma}{1-\sigma}}; E_0^K = \frac{Y_0}{K_0} \left(\frac{1}{\pi_0} \right)^{\frac{\sigma}{1-\sigma}} \quad (25)$$

Note that at $t = t_0$, $e^{\gamma_i(t-t_0)} = 1$. This ensures that at the common fixed point the factor shares are not biased by the growth of factor efficiencies but are just equal to the distribution parameters π_0 and $1 - \pi_0$.

Inserting equation (24) and the normalized values (25) into (23) leads to a normalized CES function that can be rewritten in the following form that again resembles the ACMS variant:

$$Y = \left[\pi_0 \left(\frac{Y_0}{K_0} \cdot e^{\gamma_K(t-t_0)} \cdot K_t \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \left(\frac{Y_0}{N_0} \cdot e^{\gamma_N(t-t_0)} \cdot N_t \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (26)$$

or equivalently,

$$Y = Y_0 \left[\pi_0 K_0^{\frac{\sigma}{\sigma-1}} \left(K_t \cdot e^{\gamma_K(t-t_0)} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) N_0^{\frac{\sigma}{\sigma-1}} \left(N_t \cdot e^{\gamma_N(t-t_0)} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (27)$$

In this specification of the normalized CES function, with factor-augmenting technical progress, the growth of efficiency levels for capital and labor is now measured by growth in the expressions $K_0 e^{\gamma_K(t-t_0)}$ and $N_0 e^{\gamma_N(t-t_0)}$, respectively, and t_0 is the baseline year. Again, we see from (27) that for $t = t_0$ we retrieve $Y = Y_0$.

Special cases of (27) are the specifications used by Rowthorn (1999), Acemoglu (2002a) or Bentolila and Saint-Paul (2003), where $N_0 = K_0 = Y_0 = 1$ is implicitly assumed, or by Antras (2004) who sets $N_0 = K_0 = 1$. Blanchard (1997), Caballero and Hammour (1998) and Berthold *et al.* (2002) work with a version of (27) where in addition to $N_0 = K_0 = 1$, $\gamma_K = 0$ is also assumed so that technological change is purely labor-augmenting.

It is also worth noting that for constant efficiency levels $\gamma_N = \gamma_K = 0$ our normalized function (27) is formally identical with the CES function that Jones (2003) (p. 12) has proposed for the characterization of the ‘short term’. In his terminology, the normalization values k_0 , y_0 and π_0 are ‘appropriate’ values of the fundamental

¹⁴ Neutrality concepts associate innovations to related movements in marginal products and factor ratios. An innovation is Harrod-neutral if relative input shares remain unchanged for a given capital-output ratio. This is also called labor-augmenting since technical progress raises production equivalent to an increase in the labor supply. More generally, for $F(X_i, X_j, \dots, A)$, technical progress is X_i -augmenting if $F_{AA} = F_{X_i X_i}$.

production technology that determines long-run dynamics. This long-run production function is then considered to be Cobb–Douglas with constant factor shares equal to π_0 and $1 - \pi_0$ and with a constant exogenous growth rate. Actual behavior of output and factor inputs is thus modeled as permanent fluctuations around ‘appropriate’ long-term values. For a similar approach in which Cobb–Douglas parameter values are used to normalize a CES production function, see Guo and Lansing (2009).

2.5.2 Growth Rates in Normalized Technical Progress Functions: Time-Varying Frameworks

Following recent theoretical discussion about possible biases in technical progress (e.g., Acemoglu 2002a), it is not clear that growth rates of technical progress components should always be constant. An innovation of Klump *et al.* (2007) was to allow deterministic but time-varying technological progress terms where curvature or decay terms could be uncovered from the data in economically meaningful ways. For this they used a Box and Cox (1964) transformation in a normalized context:

$$g_i(\gamma_i, \lambda_i, t, t_0) = \frac{\gamma_i}{\lambda_i} t_0 \left(\left[\frac{t}{t_0} \right]^{\lambda_i} - 1 \right), i = K, N \quad (28)$$

The curvature parameter λ_i determines the shape of the technical progress function. For $\lambda_i = 1$, technical progress functions, g_i , are the (textbook) linear specification; if $0 < \lambda_i < 1$ they are exponential; if $\lambda_i = 0$ they are log-linear and $\lambda_i < 0$ if they are hyperbolic functions in time. Note, the re-scaling of γ_i and t by the fixed point value t_0 in (28) allows us to interpret γ_N and γ_K directly as the rates of labor- and capital-augmenting technical change at the fixed-point period.

Asymptotically, function (28) would behave as follows:

$$g_i(\gamma_i, \lambda_i, t, t_0) = \begin{cases} \lim_{t \rightarrow \infty} g_i = \infty & \text{for } \lambda_i \geq 0 \\ \lim_{t \rightarrow \infty} g_i = -\frac{\gamma_i}{\lambda_i} t_0 & \text{for } \lambda_i < 0 \end{cases}$$

$$\frac{\partial g_i}{\partial t} = \gamma_i \left(\frac{t}{t_0} \right)^{\lambda_i - 1} \Rightarrow \begin{cases} \frac{\partial g_i}{\partial t} = \gamma_i & \text{for } \lambda_i = 1 \\ \lim_{t \rightarrow \infty} \frac{\partial g_i}{\partial t} = 0 & \text{for } \lambda_i < 1 \end{cases}$$

This framework allows the data to decide on the presence and dynamics of factor-augmenting technical change rather than it being imposed *a priori* by the researcher. If, for example, the data supported an asymptotic steady state, this would arise from the estimated dynamics of these curvature functions [i.e., labor-augmenting technical progress becomes dominant (linear), that of capital absent or decaying].

In addition, as McAdam and Willman (2013) pointed out, the framework also allows one to nest various strands of economic convergence paths towards the steady state. For instance, the combination,

$$\gamma_N > 0, \lambda_N = 1; \gamma_K = 0, \lambda_K = 0 \quad (29)$$

coupled with the assumption, $\sigma \gg 1$ corresponds to that drawn upon by Caballero and Hammour (1998) and Blanchard (1997), in explaining the decline in the labor income share in continental Europe.

Another combination speculatively termed ‘Acemoglu-Augmented’ Technical Progress by McAdam and Willman (2013) can be nested as

$$\gamma_N, \gamma_K > 0; \lambda_N = 1, \lambda_K < 1 \quad (30)$$

where $\sigma < 1$ is more natural.

Consider two cases within (30). A ‘weak’ variant, $\lambda_K < 0$, implies that the contribution of capital augmentation to TFP is bounded with its growth component returning rapidly to zero; a ‘strong’ case, where $0 < \lambda_K < 1$, capital imparts a highly persistent contribution with (asymptotic convergence to) a zero growth rate. Both cases are asymptotically consistent with a BGP, where output growth converges to that of labor-augmenting technical progress, γ_N , plus the rate of growth of the labor force. Accordingly, the interplay between $|\gamma_N - \gamma_K|$ and λ_K, λ_N can be considered sufficient statistics of BGP divergence. Normalization, moreover, makes this kind of classification quite natural since we are looking at biases in technical progress relative to some representative point.

2.6 An Aside: The Substitution Elasticity and Factor Income Shares

Normalization allows us to have a valid idea of the reference points for factor income shares and the bias of non-neutral technical change – in both cases, biases relative to a benchmark or reference point. Regarding the former, we know (e.g., McAdam and Willman 2013) that factor shares vary over time. Indeed such variation – as La Grandville (2009) notes – has been a major motivation for moving to more flexible functional forms for production. We now explain the link between them.

Non-neutral technical change will matter in so far it influences developments in output, relative prices, factor intensities, income shares and cost pressures. Movements in these variables affect the inter-temporal decisions of consumers and firms. Some indications of the key role played by factor substitution can be gauged from the following. Assuming competitive markets and profit maximization, relative factor income shares are:

$$\Theta_t = \frac{r_t K_t}{w_t N_t} = \frac{\pi_0}{1 - \pi_0} \left(\frac{\Gamma_t^K K_t / K_0}{\Gamma_t^N N_t / N_0} \right)^{\frac{\sigma-1}{\sigma}} \quad (31)$$

where $\Gamma_t^K = e^{\gamma^K(t-t_0)}$ etc.

It is straightforward to show that the effect of technical bias and capital deepening on factor income shares is related to whether factors are gross complements or gross substitutes:

$$\text{sgn} \left\{ \frac{\partial \Theta}{\partial (K/N)} \right\}, \text{sgn} \left\{ \frac{\partial \Theta}{\partial (\Gamma_t^K / \Gamma_t^N)} \right\} = \text{sgn} \{ \sigma - 1 \} \quad (32)$$

Hence, an increase in factor J -augmenting ($J = K, N$) technical change ‘favors’ factor J (i.e., implying $\frac{\partial F_J / \partial \Gamma^J}{\partial F_I / \partial \Gamma^I} = 1, J \neq I$, and raising J ’s income share for given factor proportions) if factors are gross substitutes ($\sigma > 1$). The effects reverse if factors are gross complements.

Thus, it is only in the gross-substitutes case that a factor J -augmenting change in

technology is J -biased. Naturally, the relations between the substitution elasticity, technical bias and factor shares evaporate under Cobb–Douglas: factor income shares are constant and relative factor prices are purely determined by capital deepening.

Equation (31) illustrates the impact of technology shocks on factor payments depends on the substitution elasticity and the factor bias of the shock. This influences the dynamic response of interest and wages (and hence hours) to technology shocks. Note, though, statement (32) defines factor *demand* reactions to technology changes. They therefore abstract from labor-supply reactions (although we will return to this in Section 5).

3. The Elasticity of Substitution as an Engine of Growth

Although one of the first references to a CES structure of aggregate production appears in the Solow (1956) paper, it had been for a long time impossible to answer the question of what effect the size of the substitution elasticity has on the steady-state values in the standard neoclassical growth model. Common sense would certainly suggest that easier factor substitution – via helping to overcome diminishing returns – should lead to a higher level of development. But a formal proof of this conjecture seemed for a long time out of reach. In fact, when Harbrecht (1975) tried to answer this question with the help of a (implicitly normalized) David and van de Klundert (1965) CES variant, he found the contrary result! His analysis was, of course, biased by the dependency of the distribution and efficiency parameters on the elasticity of substitution, when the CES function is not correctly normalized.

Already some years earlier, as mentioned in Section 2.4, Kamien and Schwartz (1968) had presented a proof of the central relationship between the substitution elasticity and output but only for the special case in which the baseline values for K and N were equal. Their proof is based on the General Mean property of the CES function, which had already been recognized by ACMS.

A General Mean of order p is defined as

$$M(p) = \left[\sum_{i=1}^n f_i x_i^p \right]^{\frac{1}{p}} \quad (33)$$

where x_1, \dots, x_n are positive numbers (of the same dimension) and where the weights f_1, \dots, f_n sum to unity. Special cases of the General Mean are the arithmetic, the geometric and the harmonic means where the order p would be 1, 0 and -1 , respectively. If p tends to $-\infty$, the mean becomes the minimum of the numbers (x_1, \dots, x_n) .

One of the most important theorems about a General Mean is that it is an increasing function of its order (Hardy *et al.* 1934, p. 26f; Beckenbach and Bellman 1961, pp. 16–18; see also the proof in La Grandville 2009, pp. 111–113). More exactly, it says that the mean of order p of the positive values x_i with weights f_i is a strictly increasing function in p unless all the x_i are equal. With the two factors K and N (and implicit normalization $K_0 = N_0$) this leads to the following statement:

Enlargement of the elasticity of substitution results in an increase in output from every combination of factors except that for which the capital labor ratio is equal to

one. (Kamien and Schwartz 1968, p. 12)

Of course, this result can be generalized provided that all numbers have the same dimension which is precisely achieved by normalizing numbers of different dimensions.

La Grandville (1989b) developed a graphical representation of normalized CES structures. He demonstrated that the general relationship between the elasticity of substitution and the level of development is usually positive. Moreover, when there are two factors of production, numerical results suggest that the function has a *single* inflection point (La Grandville and Solow, 2006): in other words, between its limiting values, $\lim_{p \in (-\infty, \infty)} M(p)$, the function $M(p)$ is first convex then concave. For typical production-function weights (i.e., $f_1 = 0.4$; $f_2 = 1 - f_1$) that inflection point occurs around $p \approx 0$ (i.e., the Cobb–Douglas neighborhood).

This means that within some relevant interval around that even small perturbations of the substitution elasticity (however such a change may be implemented) might have extremely large implications for an economy. In short, raising your elasticity of substitution can raise your growth rate and its effect may be potentially even larger than that traditionally studied in the case of (equivalent percentage) improvements in the savings rate and/or technical progress (such reasoning is reflected in the third quote that started our paper).

The formal proof for the conjecture was then presented by Klump and de La Grandville (2000), based on a very general normalized CES production function. An alternative proof is presented in Klump and Irmen (2009) who also deal with normalized CES functions in a Diamond-type version of the neoclassical growth model. It distinguishes efficiency and distribution effects of changes in the elasticity of substitution which can work in different directions if not all individuals have the same savings pattern so that redistribution matters. The interaction of both effects creates an acceleration effect for capital accumulation which can have a positive or a negative effect on the steady state. It can be shown, however, that even in this setting a higher elasticity of substitution leads to a higher steady-state level as long as the efficiency effect dominates the distribution effect, which is the most likely case.

Klump and Preissler (2000) extend the analysis of the standard neoclassical growth model with a normalized CES production function by calculating the effect of the size of the elasticity of substitution on the speed of convergence towards the steady state. Earlier studies of this problem, for example, Ramanathan (1975), which were not considering normalization had not derived convincing results. With an explicitly normalized CES production function, it is possible to show that an increase in the elasticity of substitution reduces the speed of convergence if the steady-state value of capital intensity is higher than its baseline value (which seems the most likely case).

Klump (2001) presents the analysis of a Ramsey type (intertemporal optimizing) growth model with a normalized CES production function. He is able to prove that as long as the steady-state value of the capital intensity is higher than its baseline value the comparative-static effect of a change in the elasticity of substitution on the steady state is strictly positive. The result were only recently reproduced by Xue and Yip (2013) using a different approach. For the effect of the elasticity of substitution on the

speed of adjustment the same results as in the Solow model can be derived in the Ramsey model (see Klump and Saam 2008). This result holds irrespective of the value of the elasticity of substitution, whereas Ventura (1997) making use of an implicitly normalized CES production function could only generate meaningful results for $\sigma < 1$.

Summing up, an increase in σ increases the steady-state level of production and capital intensity while lengthening the convergence time to the new steady state. From the standpoint of short run growth, this leaves open the question of whether growth in the short run will increase or decrease relative to an initially lower σ comparative value.

Temple (2008) has criticized the use of normalized CES functions for calculating convergence effects of a higher factor substitution because of an unclear economic meaning of the chosen baseline value for the capital intensity. However, as has been clarified by Klump and Saam (2008) the essence of normalization does not consist in the arbitrary choice of baseline values but in forcing the researcher to give an explicit statement about the relationship between baseline and steady-state (*ss*) values. As growth models are generally motivated by the idea that labor is relatively scarce in the steady state it seems reasonable to normalize such that $k_{ss} > k_0$. In addition, in a growing economy, it is always feasible to assume capital intensity would be below the steady state, whereas values above the steady state raise the question of how the economy has found itself in such a starting position with surplus capital stock.

The setting may be different in the business-cycle literature, where fluctuations around the (typically zero growth) steady state are studied. In this case it makes sense to use steady-state values as normalization parameters (Guo and Lansing 2009; Cantore *et al.* 2010, hereafter CLMW (2010)).

Finally, Irmen (2011) is able to show in an endogenous growth framework with a normalized CES production function that the steady-state growth rate of output per worker increases with the elasticity of substitution. The efficiency effect induced by a higher degree of factor substitution makes innovation investments more profitable that raise permanently the productivity of labor. All analysis using normalized CES production functions confirm that the elasticity of substitution is among the most powerful determinants of growth. La Grandville (2009), 2012) suggests that changes in the elasticity of substitution have a much higher effect on social welfare than changes in the rate of technical progress – see chapter 13, pp. 316-319, where the author compares the ratio of sensitivities of a given value function, V (e.g., savings or consumption flows over time), to a change in the elasticity of substitution and to a change in the rate of (Harrod-neutral) technical progress: $e_{V,\sigma} / e_{V,\gamma_N}$.

4. Estimated Normalized Production Function

Previous sections of our paper introduced the concept of normalization and largely its importance in theoretical analysis. Here we discuss how the idea of normalization should be applied in *empirical* analysis and, more importantly, whether it makes the estimation of the parameters of the CES production function easier and more robust? We show that its merits are strong especially if system approach (containing

cross-equation restrictions) is used. In this context the scepticism on the proper identification of the elasticity of substitution and technical progress from each other aroused by the famous Diamond–McFadden impossibility theory largely loses its practical importance. In fact, we argue that a general factor-augmenting specification results in markedly less biased estimates of the elasticity of substitution parameter than imposing *a priori* neutrality constraints. In the context of single equation approach, though, normalization is of lesser use.

An added problem,¹⁵ however, is that often the predictions of different elasticity and technical change combinations can have similar implications for variables of interest, such as factor income shares and factor ratios. Notwithstanding, whether factor income movements are driven by high or low substitution elasticities and with different combinations of technical change is profoundly important in terms of their different implications for, for example, growth accounting, inequality, calibration in business-cycle models, public policy issues etc.

By way of illustration, Tables 1 and 2 present an overview of empirical results obtained for the elasticity of substitution. We concentrate on the results from time-series or panel studies on aggregate data. In the case of the United States, which has been widely studied, it is possible to find values of the elasticity of substitution above unity (with Harrod-neutral technical progress), at unity (with Hicks-neutral progress) and below unity (with Hicks-neutral progress and with technical progress augmenting both factors). The situation for other countries is no better; for Germany, values of above, below and at unity have been estimated. Using information about the degree of factor substitution from other sources does not resolve this puzzle, either. It has been recognized, for example, by Lucas (1969) that older time-series studies for the United States have generally provided lower estimates than cross-section studies that were supportive of the Cobb–Douglas function. More recent cross-section analysis based on micro data that were used to estimate the relationship between business capital formation and user costs (e.g., Chirinko *et al.* 1999) estimates very low elasticities of substitution ranging from 0.25 to 0.40. A drawback of these kinds of studies, however, is their inability to quantify any growth rate(s) of technical progress.

That there should be diversity in production function estimates – even for countries whose data properties are relatively stable and well-understood – is not surprising. It doubtlessly reflects the familiar trapdoor of empirical pitfalls: data quality; *a priori* modeling choices (such as whether to test for certain types of factor neutrality or impose them); the performance of various estimators (e.g., single equation, systems) and algorithms; as well as more prosaic data problems (e.g., outliers, uncertain auto-correlation, structural breaks, quality improvements, measurement errors etc.).

At a simple level, normalization removes the problem that arises from the fact that labor and capital are measured in different units – although as we have seen its importance goes well beyond that. Under Cobb–Douglas, normalization plays no role since, due to its multiplicative form, differences in units are absorbed by the scaling

¹⁵ See the discussion in Leon-Ledesma *et al.* (2010b) and possible observational equivalence in examining income share developments and inferring the associated bias in technical progress.

constant. The CES function, by contrast, is highly non-linear, and so, unless correctly normalized, excluding technical progress, out of its three key parameters – the efficiency parameter, the distribution parameter, the substitution elasticity – only the latter is ‘deep’. The other two parameters turn out to be affected by the size of the substitution elasticity and factor income shares.

Table 1. Empirical Studies of Aggregate Elasticity of Substitution and Technological Change in the US.

Study	Sample ^a	Assumption on Technological Change	Estimated Elasticity of Substitution: σ	Estimated Annual Rate Of Efficiency Change		
				Neutral: $\gamma_N = \gamma_K$	Laboraaugmenting: γ_N	Capital- Augmenting: γ_K
Arrow <i>et al.</i> (1961)	1909-1949	Hicks-Neutral	0.57	1.8	–	–
Kendrick and Sato (1963)	1919-1960	Hicks-Neutral	0.58	2.1	–	–
Brown and De Cami (1963)	1890-1918	Factor Augmenting	0.35	Labor saving ($\gamma_N - \gamma_K = 0.48$)		
	1919-1937		0.08	Labor saving ($\gamma_N - \gamma_K = 0.62$)		
	1938-1958		0.11	Labor saving ($\gamma_N - \gamma_K = 0.36$)		
	1890-1958		0.44	?		
David and van de Klundert (1965)	1899-1960	Factor Augmenting	0.32	–	2.2	1.5
Bodkin and Klein (1967)	1909-1949	Hicks-neutral	0.50-0.70	1.4-1.5	–	–
Wilkinson (1968)	1899-1953	Factor Augmenting	0.50	–	Labor saving ($\gamma_N - \gamma_K = 0.51$)	
Sato (1970)	1909-1960	Factor Augmenting	0.50 – 0.70	–	2.0	1.0
Panik (1976)	1929-1966	Factor Augmenting	0.76	–	Labor saving ($\gamma_N - \gamma_K = 0.27$)	
Berndt (1976)	1929-1968	Hicks-neutral	0.96-1.25	?	–	–
Kalt (1978)	1929-1967	Factor Augmenting	0.76	–	2.2	0.01
Antràs (2003)	1948-1998	Hicks-neutral	0.94-1.02	1.14	–	–
		Factor-augmenting	0.80	–	Labor saving ($\gamma_N - \gamma_K = 3.15$)	
Klump <i>et al.</i> (2007b) ^b	1953-2002	Factor-augmenting	0.7	–	–	–
León-Ledesma <i>et al.</i> (2010)	1960- 2004	Factor-augmenting	0.60-0.70	–	1.60	0.70

Notes:

^a All studies are estimated on annual frequency data except León-Ledesma *et al.* (2010b) which is quarterly. Although, to aid comparability, we annualized their estimates for technical change in the table.

^b We do not report technical change estimates for Klump *et al.* (2007b) since they estimate with structural breaks in their technical progress terms. For reasons of space, the reader is referred to León-Ledesma *et al.* (2010a) for the original references.

Table 2. Empirical Studies of Aggregate Elasticity of Substitution in Selected Other Countries.

Study	Countries	Sample (Frequency)	Assumption For Technological Change	Estimated Elasticity Of Substitution: σ
Lewis and Kirby (1988)	Australia	1967-1987 (Weekly)	Hicks-Neutral	0.78
Easterly and Fischer (1995)	Soviet Union	1950-1987 (Annual)	Hicks-Neutral	0.37
Andersen <i>et al.</i> (1999)	Panel of 17 OECD countries	1966-1996 (Annual)	Hicks-Neutral	1.12
Bolt and van Els (2000)	Austria	1971-1996 (Quarterly)	Hicks-Neutral	0.24
	Belgium			0.78
	Germany			0.53
	Denmark			0.61
	Spain			1
	Finland			0.34
	France			0.73
	Italy			0.52
	Netherlands			0.27
	Sweden			0.68
	UK			0.6
	US			0.82
	Japan			0.3
Duffy and apageorgiou (2000)	Panel of 82 developed and developing countries	1960-1987 Annual)	Hicks-Neutral	1.4
Ripatti and Vilminen (2001)	Finland	1975-1999 (Quarterly)	Factor-Augmenting	0.6
Willman (2002)	Euro area	1970-1997 (Quarterly)	Solow-Neutral	0.95-1.05
McAdam and Willman (2004)	Germany	1983-1999 (Quarterly)	Hicks-Neutral	0.7, 1, 1.2
Berthold <i>et al.</i> (2002)	US, Germany, France	1970-1995 (Semi-Annual)	Harrod-Neutral	1.15, 1.45, 2.01
Bertolila and Saint-Paul (2003)	13 industries in 12 OECD countries	1972-1993	Harrod-Neutral	1.06
McAdam and Willman (2004a)	Germany	1983-1999 (Quarterly)	Hicks-Neutral	0.7, 1, 1.2
Klump <i>et al.</i> (2007b)	Euro Area	1970-2003 (quarterly)	Factor-Augmenting	0.7
Luoma and Luoto (2010)	Finland	1902-2004 (annual)	Factor-Augmenting	0.5

Notes: For reasons of space, the reader is referred to Klump *et al.* (2007b) for most of these original references.

If one compares the explicitly normalized with the implicitly normalized function, as before, that is,

$$Y_t = C \left[\pi \left(\Gamma_t^K K_t \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi) \left(\Gamma_t^N N_t \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

$$Y_t = Y_0 \left[\pi_0 \left(\frac{\Gamma_t^K K_t}{\Gamma_0^K K_0} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \left(\frac{\Gamma_t^N N_t}{\Gamma_0^N N_0} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

we may be unsure as to where the estimation benefits of normalization derive. After all, both equations contain the same number of parameters. In fact, the latter equation seemingly *adds* complexity by incorporating normalized reference points into the

estimation (the empirical choice of the normalization point is a particular aspect discussed in Section 4.2).

The answer as to why normalization should improve matters empirically reflects the following. The distribution and efficiency parameters (respectively, Y_0 ; π_0 and C ; π) can now either be imposed prior to estimation or at least have a deep interpretation in terms of the data (i.e., the representative capital income share). Effectively normalization allows us to reduce the number of freely estimated parameters by two.

This follows straightforwardly from our earlier analysis. In the implicitly normalized formulation the parameters C and π above have no clear theoretic or empirical meaning. Instead, they are composite parameters conditional on, besides the selected fixed points, the elasticity of substitution (re-stating equations (15) and (16)):

$$C(\sigma, \cdot) = Y_0 \left[\frac{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}}{r_0 K_0 + w_0 N_0} \right]^{\frac{\sigma}{\sigma-1}}$$

$$\pi(\sigma, \cdot) = \frac{r_0 K_0^{1/\sigma}}{r_0 K_0^{1/\sigma} + w_0 N_0^{1/\sigma}}$$

The additional merit in using the normalized instead of the implicitly normalized form is that all parameters have a clear empirical correspondence. In particular, the distribution parameter is identified as the capital income share of total factor income at the fixed point. Hence, a suitable choice for the fixed point may alleviate the estimation of the deep parameters and, to repeat, makes the estimated production function suitable, for example, for subsequent comparative-static analysis.¹⁶

Table 3 presents some consistent sets of (deterministic) initial values for generating data and the implied ranges of the true values of C and π for $\sigma \in [0.2, 1.3]$. In the first row we assume $K_0 = N_0 = 1$. This allows us to solve Y_0 from the first row – with initial values of $\Gamma_0^K = \Gamma_0^N = 1$.

In fact this represents a special case because indexing by the point of normalization equaling one is neutral implying that the true value of $C = 1$ and $\pi = \pi_0 = r_0 = 0.3 \forall \sigma$ (this, in turn, implies solving the normalized real wage rate as $\frac{(1-\pi_0)Y_0}{N_0}$). In this special case it does not matter if the same initial values of parameters are used, whether the system is estimated in normalized or implicitly normalized form.

In all other cases, however, this is not so. To illustrate, in these other cases we have adjusted the initial conditions for output to make them consistent with an initial (and arguably more reasonable) value for r (the real user cost of capital) equal to 5%. The sample average normalization insulates the normalized system from the effects of changes in initial values in generating the data but the true values of composite parameters C and π vary widely: $C \in [0.23, 0.79]$, $\pi \in [0.23, 0.99]$. Thus, we confirm that the actual income distribution of the data is completely unrelated to the true value of π .

¹⁶ We used the word ‘deep’ when we discuss the C and π parameters. By deep, we meant that it is not dependent on any parameter other than itself. In the implicitly normalized case, the parameters of the production function (the efficiency and distribution parameter) are functions of σ (except for the counterfactual case of $K = N = 1$), and are therefore not deep.

This illustrates the difficulty that a practitioner faces when trying to estimate implicitly normalized forms since the actual data scarcely give any guidelines for appropriate choices for the initial parameter values of C and π .

Table 3. Consistent Normalization Values.

N_0	π_0	r_0	K_0	$Y_0^* = Y_0 = \frac{r_0}{\pi_0} K_0$	$w_0 = \frac{(1-\pi_0)Y_0}{N_0}$	C		π	
						when $\sigma = 1.3$	when $\sigma = 0.2$	when $\sigma = 1.3$	when $\sigma = 0.2$
1	0.3	0.3	1	1	0.7	1		0.4	
1	0.3	0.05	5	0.833	0.583	0.546	0.225	0.228	0.996
1	0.3	0.05	8	1.333	0.933	0.788	0.225	0.210	0.999

Notes: C and π in the final two columns are calculated according to equations (15) and (16) for $\sigma \in [0.2, 1.3]$. Outside of the “special case” note the following partial derivatives showing how ceteris paribus changes in initial values

change these last two parameters, C and π :

$$C_\sigma, C_{Y_0}, C_{K_0}, C_{w_0} > 0, C_{N_0} < 0$$

$$\pi_\sigma, \pi_{N_0} < 0, \pi_{K_0}, \pi_{w_0} > 0$$

As Leon-Ledesma *et al.* (2010a) have documented, that results in serious estimation problems. They estimated normalized and implicitly normalized forms where in the latter case the initial parameter values for C and π are selected randomly from their given range such as in Table 3. When \hat{C} and $\hat{\pi}$ substantially depart from their true, theoretical values, there are significant and quantitatively important biases in the estimated substitution elasticity and technical change.

4.1 Estimation Forms

The recognition of normalization says nothing specifically about the way production and production technology should be estimated and how normalization impacts those estimation choices. Typical estimation forms found in the literature include: the non-linear CES production function; the linear first-order conditions of profit maximization; linear approximation of the CES function and ‘system’ estimation incorporating the production function and the first-order conditions.¹⁷

To proceed let us express the CES function, equation (27), in log form:

$$\log\left(\frac{Y}{Y_0}\right) = \frac{\sigma}{\sigma-1} \log\left[\pi_0 \left(\frac{e^{\gamma_K(t-t_0)} K_t}{K_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\pi_0) \left(\frac{e^{\gamma_N(t-t_0)} N_t}{N_0}\right)^{\frac{\sigma-1}{\sigma}}\right] \quad (34)$$

From this we can derive the marginal profit-maximization conditions,¹⁸

¹⁷ We confine ourselves to constant-returns production functions. This is largely done to be consistent with much of the aggregate evidence (e.g., Basu and Fernald 1997).

¹⁸ Given that the real user cost and real interest rate can be sometimes negative in historical samples (particularly in the 1970s), the user cost conditions is usually expressed in levels rather than logarithms. Note, the last two conditions in some estimation cases are merged in many papers:

$$\log\left(\frac{K_t}{N_t}\right) = \alpha_i + \sigma \log\left(\frac{w_t}{r_t}\right) + (\gamma_N - \gamma_K)(1-\sigma)t$$

$$\log\left(\frac{K_t}{N_t}\right) = \alpha_j - \frac{\sigma}{1-\sigma} \log\left(\frac{r_t K_t}{w_t N_t}\right) + (\gamma_N - \gamma_K)t$$

$$\log(r) = \underbrace{\log\left(\pi_0 \frac{Y_0}{K_0}\right) + \frac{1}{\sigma} \log\left(\frac{K_0}{Y_0}\right)}_{\alpha_r} + \frac{1}{\sigma} \log\left(\frac{Y}{K}\right) + \frac{\sigma-1}{\sigma} (\gamma_K (t-t_0)) \quad (35)$$

$$\log(w) = \underbrace{\log\left((1-\pi_0) \frac{Y_0}{N_0}\right) + \frac{1}{\sigma} \log\left(\frac{N_0}{Y_0}\right)}_{\alpha_w} + \frac{1}{\sigma} \log\left(\frac{Y}{N}\right) + \frac{\sigma-1}{\sigma} (\gamma_N (t-t_0)) \quad (36)$$

Where, as before, γ_N and γ_K are the respective growth rates of labor and capital augmenting technical progress. Equations (35) and (36) represent the first-order conditions with respect to capital and labor, respectively.

Estimation of production and technology parameters based on the first-order conditions and other single-equation approaches is hampered by the fact that they only admit estimates of technical progress terms contained by their presumed FOC choice (in that sense any *bias* in technical progress is, by definition, not separately identifiable). This apparent drawback is presumably (in the minds of the researcher at least) compensated by their tractable form and linearity. Accordingly, these forms are common (more common, for instance, than direct non-linear CES estimation): For example, equation (35) has been widely used in the investment literature (e.g., Caballero 1994) and (36) was the form used by ACMS amongst many others.

A notable feature of the above three equations is that if estimated in *single*-equation mode, the normalization points (denoted by the curly lower brackets) are absorbed by the respective constants, α_r and α_w . Thus, from an estimation stand point, it is only when the non-linear CES function is estimated directly or where the system approach is used, does formal normalization play an explicit empirical role.

Another possible vehicle of estimation is the Kmenta (1967) approximation (which became an important, if apparently unacknowledged, pre-cursor to the translog form). This is a Taylor-series expansion of the log CES production function around $\sigma = 1$.¹⁹

$$y_t = \pi_0 k_t + \lambda k_t^2 + \underbrace{\pi_0 \left[1 + \frac{2\lambda}{\pi_0} k_t \right] \gamma_K \tilde{t} + (1-\pi_0) \left[1 - \frac{2\lambda}{1-\pi_0} k_t \right] \gamma_N \tilde{t} + \lambda [\gamma_K - \gamma_N]^2 \cdot \tilde{t}^2}_{t f p} \quad (37)$$

where $\tilde{t} = t - t_0$, $y_t = \log[(Y/Y_0)/(N/N_0)]$, $k_t = \log[(K/K_0)/(N/N_0)]$, $t f p = \text{Log}(T F P)$ and $\lambda = \frac{(\sigma-1)\pi_0(1-\pi_0)}{2\sigma}$. Equation (37) shows that the output–labor ratio can be decomposed into capital deepening and technical progress, weighted by factor shares and the substitution elasticity (where $\text{sgn}(\lambda) = \text{sgn}(\sigma - 1)$ and $\lim_{\lambda \in [0, \infty)} \lambda \in [-\infty, 1/2\pi_0(1-\pi_0)]$). In addition, (37) shows that, when $\sigma \neq 1$ and $\gamma_K \neq \gamma_N > 0$, additional (quadratic²⁰) curvature is introduced into the estimated production function.

With the predetermined normalization point, the advantage of (37) over the Kmenta approximation of the implicitly normalized CES is – as usual – that, since all variables appear in indexed form, the estimates are invariant to a change in units of measurement. Another advantage is that in the neighborhood of the normalization

¹⁹ Linearization around a unitary substitution is algebraically the most convenient form, as can be easily verified.

²⁰ This is quadratic or higher depending on the order of the approximation.

point (i.e., $K_t = K_0$, $N_t = N_0$, $\pi = \pi_0$) and without σ deviating ‘too much’ from unity, as the approximation also assumes, the terms including the normalized capital intensity and multiplying linear trend have only second-order importance and, without any significant loss of precision, can be dropped, yielding,

$$y = \pi_0 k_t + \lambda k_t^2 + \underbrace{[\underbrace{\pi_0 \gamma_K + (1 - \pi_0) \gamma_N}_{\theta} \tilde{t} + \underbrace{\lambda [\gamma_K - \gamma_N]^2 \tilde{t}^2}_{\epsilon}]}_{TFP} \quad (38)$$

Estimation of equation (38) yields four parameters, π_0 , $\hat{\lambda}$, $\hat{\theta}$, $\hat{\epsilon}$, for four primitives, π_0 , σ , γ_K , γ_N . Using π_0 allows us to identify σ from composite parameter λ , that is, $\sigma = (1 - \frac{2\hat{\lambda}}{\pi_0(1-\pi_0)})^{-1}$. However, without *a priori* information on which one of two technical progress components dominates and, in addition, that the signs of estimates λ and σ are (or are constrained to be) the same, one cannot identify γ_K and γ_N . This leads to the following weak identification result: for $\gamma_N > \gamma_K$ we obtain $\gamma_N = \hat{\theta} + \pi_0 \sqrt{\frac{\hat{\epsilon}}{\hat{\lambda}}}$ and $\gamma_K = \hat{\theta} - (1 - \pi_0) \sqrt{\frac{\hat{\epsilon}}{\hat{\lambda}}}$ and for $\gamma_N < \gamma_K$ we obtain, $\gamma_N = \hat{\theta} - \pi_0 \sqrt{\frac{\hat{\epsilon}}{\hat{\lambda}}}$ and $\gamma_K = \hat{\theta} + (1 - \pi_0) \sqrt{\frac{\hat{\epsilon}}{\hat{\lambda}}}$.

Given this, even under the helpful environment of normalization, we can say that although the Kmenta approximation can be used to estimate σ , it cannot effectively identify the direction of the biased technical change.²¹ However, the approximation is a useful vehicle to, *ex post*, calculate TFP.

4.2 The Point of Normalization – Literally!

To be empirically operational, *the point of normalization* must be defined (i.e., what these Y_0 , K_0 are in practice). If the estimation data were deterministic, this would be unproblematic: every sample point would be equally suitable for the point of normalization. For instance, in theoretic settings, the normalization point is often calibrated around the non-stochastic steady state (e.g., Klump and Saam 2008; CLMW 2010).

However, since actual data are inevitably stochastic (and the intensity with which factors are utilized is also unobserved), this convenience does not carry over because the production function does not hold exactly in any sample point. Therefore, to diminish the size of cyclical and stochastic components in the point of normalization, an appealing procedure is to calibrate the normalization point in terms of sample averages for the underlying variables – typically geometric averages for growing variables (such as factors of production) and arithmetic ones for approximately stationary variables (e.g., factor income shares, the real interest rate and user cost).

The non-linearity of the CES function, however, in turn, implies that the sample

²¹ The Kmenta approximation, both empirically and in terms of general identification, has enjoyed limited success (see Kumar and Gapinski 1974; Thursby 1980; Leon-Ledesma *et al.* 2010a).

average of production need not exactly coincide with the level of production implied by the production function with sample averages of the right hand variables *even with a deterministic DGP*. To circumvent this problem, Klump *et al.* (2007) introduced an additional parameter ξ whose expected value is around unity (which we might call the normalization constant²²). Hence, we can define $Y_0 = \xi \bar{Y}$, $K_0 = \bar{K}$, $N_0 = \bar{N}$, $\pi_0 = \bar{\pi}$ and $t_0 = \bar{t}$ where the bar refers to the respective sample average (geometric or, as in the last two, arithmetic).

An advantage of the explicitly normalized system over the implicitly normalized system, in turn, is that the distribution parameter $\bar{\pi}$ has a clear data-based interpretation. Therefore, it can either be fixed prior to estimation or, at least, the sample average can be used as a very precise initial value of the distribution parameter. Likewise, a natural choice for the initial value of normalization constant, ξ , is one. Estimated values of these two parameters should not deviate much from their initial values without casting serious doubts on the reasonableness of estimation results. In the implicitly normalized case, by contrast, no clear guidelines exist in choosing the initial values of distribution parameter π and efficiency parameter C . This is especially the case if the function that the estimation method optimizes contains several local optima, as demonstrated to be a concrete problem in analyzing both real data (Klump *et al.* 2007) and Monte Carlo generated data (Leon-Ledesma *et al.* 2010b). Hence, in the context of non-linear estimation this may imply a well-defined advantage of the explicitly normalized over the implicitly normalized system.

5. Normalization in Growth and Business-Cycle Models

Production functions are a key part of business-cycle and growth macro-models. Over the last few decades, the two main competing models in the macro profession have been the RBC (real business cycle) model and the NK (New Keynesian) model. Both imply relatively tight, theory-led dynamics and are typically furnished with a rich number of stochastic shocks which displace the agent from his optimal plans. The standard RBC model is a variant of the representative agent neoclassical model, where business cycles are due to non-monetary sources (primarily, changes in technology). The NK one supplements that with various real and nominal rigidities to better capture the data.

However, what both models tend to share in practice is a focus on Cobb–Douglas aggregate production. This is especially puzzling given that such models tend to be motivated as *business-cycle* frameworks. Yet over business-cycle frequencies one might precisely expect relatively little (and presumably below unitary) factor substitutability as well as the presence of *non-neutral* technical change to capture factor income share developments. By introducing and assessing non-unitary production forms, the potential for a better understanding of technology and policy transmission and for a richer decomposition of historical time-series is likely to be

²² Only in the log-linear case of Cobb–Douglas would one expect ξ to exactly equal unity. Hence, in choosing the sample average as the point of normalization we lose precision because of the CES’s nonlinearity. If, alternatively, we choose the sample mid-point as the normalization point, we should also lose because of stochastic (and in actual data, cyclical) components that would also imply non-unitary ξ .

considerable.

The introduction of normalized technologies in simple business-cycle models is relatively straightforward and can be illustrated using the canonical RBC model. The model is relatively well known and can therefore be introduced compactly. The standard model with CES production technology in the supply side would be given by:

$$\Lambda_t = \beta \mathbb{E}_t \{ \Lambda_{t+1} [1 + r_{t+1} - \delta] \} \quad (39)$$

$$w_t = \nu \frac{N_t^\zeta}{\Lambda_t} \quad (40)$$

$$Y_t = Y_0 \left[\pi_0 \left(\frac{K_t}{K_0} e^{\gamma_t^K} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0) \left(\frac{N_t}{N_0} e^{\gamma_t^N} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad (41)$$

$$w_t = (1 - \pi_0) \left(\frac{Y_0}{N_0} e^{\gamma_t^N} \right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{N_t} \right)^{\frac{1}{\sigma}} \quad (42)$$

$$r_t = \pi_0 \left(\frac{Y_0}{K_0} e^{\gamma_t^K} \right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{K_t} \right)^{\frac{1}{\sigma}} \quad (43)$$

$$C_t + K_t - (1 - \delta)K_{t-1} \leq Y_t \quad (44)$$

$$\gamma_t^j = \rho_j \gamma_{t-1}^j + \eta_t^j \quad (45)$$

where Λ_t , w_t and r_t are, respectively, the marginal utility of consumption (C_t), wages and the interest rate (all expressed in real terms). Parameters β , δ and ν represent, respectively, the discount factor, the capital depreciation rate and a scaling constant. Processes γ_t^j are technology shocks – as equation (45) shows usually modeled as a stationary AR(1) process – for $j = K, N$ (i.e., capital-augmenting and labor-augmenting shocks, respectively). Equations (39) and (40) represent the household's optimal consumption and labor supply choices given, for example, the separable utility function,

$$U(C, N) = \frac{C_t^{1-\sigma_c}}{1-\sigma_c} - \nu \frac{N_t^{1+\zeta}}{1+\zeta} \quad (46)$$

where σ_c is the coefficient of relative risk aversion and ζ is the inverse of the Frisch elasticity. This particular utility function implies $\Lambda_t = C_t^{-\sigma_c}$. If the researcher wanted to simulate this model conditional on different values of the substitution elasticity, s/he would do the following:

- (i) Imposed key normalization parameters: $r_0 = \frac{1}{\beta} - 1 + \delta$, and, for some given N_0, K_0 , π_0 , solve out $Y_0 = \frac{r_0}{\pi_0} K_0$ and $w_0 = (1 - \pi_0) \frac{Y_0}{N_0}$ following Table 3 (for temporary shocks, these normalization points will be chosen to be the same as the presumed steady state);

- (ii) Reset the leisure scaling parameter ν to equate the real wage expressions in (40)

and (42), implying $v = \frac{(1-\alpha_0)r_0^{\sigma_c}}{(r_0-\delta\alpha_0)^{\sigma_c}}$.

In this simple way, conventional dynamic exercises can be performed on the model (e.g., examining the effect of technology shocks) which are robust to changes in the substitution elasticity.

In this vein, CLMW (2010) looked at the relationship between technology shocks and hours worked – a key controversy between NK and RBC explanation of the business cycle – by expressing both models in consistent normalized form. They showed that, depending on the value of the substitution elasticity and the source of the shock (capital- or labor-augmenting), both models could generate positive or negative hours responses (thus, largely overturning conventional wisdom on the mechanisms in the models).

6. Conclusions and Future Directions

The elasticity of substitution between capital and labor represents a key parameter in many fields of economics: for example, business-cycle and growth outcomes, income distribution, stabilization policy, labor market dynamics etc. The empirical evidence – for the United States and other developed economies – is clear in rejecting the unitary-elasticity Cobb–Douglas specification in favor of (generally below unity) CES aggregate production functions.²³ When investigating the ramifications of a non-unitary substitution elasticity, one necessarily faces the issue of normalization.

The importance of explicitly normalizing CES functions was discovered in the seminal paper by La Grandville (1989b), then further explored by Klump and de La Grandville (2000), Klump and Preissler (2000), La Grandville and Solow (2009), and first empirically implemented and investigated by Klump *et al.* (2007). Normalization starts from the observation that a family of CES functions whose members are distinguished only by different elasticities of substitution needs a common benchmark point. Since the elasticity of substitution is defined as a point elasticity, one needs to fix benchmark values for the level of production, the inputs of capital and labor and for the marginal rate of substitution, or equivalently for per-capita production, capital intensity and factor income shares.

Overall, we can say the following points.

(a) Normalization is necessary for identifying in an economically meaningful way the constants of integration which appear in the solution to the differential equation from which the CES function is derived (and thus makes it suitable for comparative-static analysis).

(b) Normalization helps to distinguish among the various functional forms, which have been developed in the CES literature and thus to choose which CES production functions are legitimate.

(c) Normalization is necessary for securing the basic property of CES production in the context of growth theory, namely the strictly positive relationship between the substitution elasticity and the output level given the CES function's representation as

²³ Duffy and Papageorgiou (2000) suggest developing countries may be better empirically represented by an above-unity aggregate substitution elasticity.

a ‘General Mean’.

(d) In situations where the researchers wish to gauge the sensitivity of results (steady-state or dynamic) to variations in the substitution elasticity, normalization is imperative.

(e) Normalization alleviates the estimation of the deep parameters of the aggregate production function, in particular the elasticity of substitution and the growth rates of factor-augmenting technical progress.

(f) Normalization is convenient when biases in the direction of technical progress are to be empirically determined, since it fixes a benchmark value for factor income shares. This is important when it comes to an empirical evaluation of changes in income distribution arising from technical progress. If technical progress is biased in the sense that factor income shares change over time the nature of this bias can be best classified with regard to a given baseline value.

That said, in our view there are at least five promising and related areas for future research on normalization:

(1) Following Jones (2005) one can regard a macroeconomic production function as a reduced form which should be derived from micro-foundations. This conjecture has been taken up by Growiec (2008) who shows that a CES production function can be linked to Weibull distributions of unit factor productivities, whereas a Cobb–Douglas function responds to Pareto distributions. It is still an open question, however, how the normalization values of the CES function can be linked in a meaningful way to the parameters of the underlying unit factor productivity distributions.²⁴

(2) A better understanding of the micro-foundations of the normalized CES function would also help to better understand reasons for possible international differences and intertemporal changes in the elasticity of substitution. Both seem to be linked to some deeper economic, social and cultural parameters as well as to the level of development measured by capital per units of labor or income per capita (Klump and de La Grandville 2000; Duffy and Papageorgiou 2000; Masanjala and Papageorgiou 2004; Mallick 2010, Wong and Yip 2010).

(3) La Grandville (2009, 2012) has suggested that an increase in the elasticity of substitution has a much stronger effect on aggregate wealth than an increase in the rate of technical progress. This suggestion can be viewed in two ways. First, in terms of comparative statics. For example, if two countries are otherwise the same but have different (though constant) substitution elasticities, one can trace the effects of that difference on their growth prospects. Second, even though the substitution elasticity empirically is (or appears to be) quite stable, such structural changes (e.g., larger internal or external markets) are possible, that make factor substitution easier and may launch some kind of ‘sigma-augmenting’ technical progress at work whose exact mechanisms are not yet understood. Kamien and Schwartz (1968) had already pointed out that changes in the elasticity of substitution have similar effects on relative factor prices and on the distribution of income as the augmentation in factor efficiency. This is obvious from expression (31) given above. As in other areas of induced technical

²⁴ Prompted by this remark in a first draft of our survey, Jakub Growiec explored the link in Growiec (2011).

change, these changes in relative factor prices and income distribution might trigger biases in the direction of technical change which are worth being analyzed in more detail.²⁵

(4) In business-cycle and growth models, the Cobb–Douglas aggregate production function is the default choice. However, the convenience/centrality of Cobb–Douglas production functions in macro is likely to be obscuring important issues. We would therefore expect that normalization –which leads to a clarified and deeper understanding of CES properties – will end up being more widely used in such models. In so doing, its use should help shed light on the propagation and decomposition of business-cycle shocks and policy transmissions.

(5) Production functions are often single-level variants, given that, in macroeconomics certainly, only two factors of production are considered. However, potentially that obscures the interactions between and within different factor categories. For example, there are high- and low-skill types of labor and different strata of capital such as equipment, software, and buildings and infrastructure. In this respect, an important departure from the aggregative framework was made in the seminal contributions of Kazuo Sato and Zvi Griliches. Sato (1967) generalized the CES production function by nesting the CES at two levels and augmenting the list of possible inputs.

A popular focus for work on the two-level CES function is on explanations of the increase in the skill premium observed in western economies during the last three decades. Does the premium reflect capital–skill complementarity (as in Griliches 1969 and Krussel *et al.* 2000)? Or can the premium can be attributed to technical change that was biased in favor of skilled workers (e.g., Katz and Murphy 1992, Acemoglu 2002b; Autor *et al.* 2008)?²⁶ Both approaches rely on particular nestings and estimated values for the elasticities of substitution between different categories of production factors and would be highly amenable to the simplification that normalization offers [see Leon-Ledesma *et al.* (2012) for some early work in that direction]. Knowledge of this might also deepen our understanding of skill differences between and within developing countries.

²⁵ Since Hicks (1932), the value of the substitution elasticity has often been seen as reflecting economic flexibility and thus deep institutional factors such as labor bargaining power, the taxation burden, degree of economic openness, the characteristics of national education system, etc. Accordingly, some view changes in the substitution elasticity as drivers of endogenous growth and as such potentially even more important than traditionally studied growth factors such as savings and technical progress (La Grandville 2009; Yuhn 1991). This is termed the ‘de La Grandville Hypothesis’ following Yuhn (1991) (i.e., the conjecture that the high growth rate of east Asian countries was due not to a higher rate of technical progress, but to a higher elasticity of substitution). Also, earlier, Solow (1956) and Pitchford (1960) showed in the neoclassical growth model that a CES function with an elasticity of substitution greater than one can generate sustained growth (even without technical progress). See also Masanjala and Papageorgiou (2004) for a theoretical model where the aggregate elasticity of substitution is endogenized and depends on the level of economic development.

²⁶ The capital–skill hypothesis gained particular currency given the sharp decline in the constant-quality relative price of equipment, for example, Gordon (1990), particularly for information and communication technologies. This decline naturally led to an uptake in usage of such capital. Given complementarity between capital and skilled labor, the faster usage of such capital increased the relative demand for skilled labor and – despite the apparent increase in the supply of such labor – the skill or wage premium relative to unskilled labor increased in a dramatic and persistent manner. On the other hand, authors such as Katz and Murphy (1992), Acemoglu (2002b) and Autor *et al.* (2008) claimed that the skill premium can be attributed to technical change that was biased in favor of skilled workers. Given that skilled and unskilled workers are gross substitutes, an increase in skilled labor efficiency led to an increase in the relative wages (and factor shares) of skilled workers.

To conclude our survey, we re-stress that production functions are ubiquitous in theoretical and empirical models, and ubiquitously Cobb–Douglas! This appears to us not only an unjustifiable simplification but also an impediment to understanding various economic phenomena. The paper by CLMW (2010) is hopefully a useful contribution in fashioning otherwise standard Real Business Cycle and New Keynesian models around a normalized CES supply side and showing the impact in terms of overturning the prediction that these separate models made for the technology–hours correlation. But this is merely one example. We would hope therefore that normalized CES functions would be an integral part of the make-up of macroeconomic models, in the same way as nominal and real rigidities have become.

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