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A model of delegation with a VaR constraint☆

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A B S T R A C T

This paper proposes a model of delegated portfolio management, in which professional fund managers face a value-at-risk (VaR) constraint. We show that the existence of the VaR constraint impairs the optimal risk sharing in both the trading and delegation stages. As a result, the VaR constraint leads household investors to take excessive risk and may cause the prices of fundamentally uncorrelated assets to be correlated.

1. Introduction

In the modern financial market, two features are salient. First, we observe a trend towards institutionalization. For example, French (2008), finds that institutional investors were accounted for more than 80% of equities ownership in the U.S. in 2007, compared to 50% in 1980. Thus, it is important to understand the behavior of professional fund managers and their impact on asset prices. Second, we often observe comovements of seemingly unrelated assets in the market, which is called contagion. Financial contagion is puzzling, and therefore receives many explanations, such as fluctuating attention (e.g. Hasler and Ornthanalai (2018)), borrowing constraints (e.g. Yuan (2005)), and institutional herding (e.g. Deng, Hung and Qiao (2018)). While institutionalization and contagion are both very important, people tend to analyze them separately. In this paper, we build an equilibrium model of delegation to analyze financial contagion.

One method for analyzing the problem of institutionalization relies on the delegation relationship between fund managers and household investors. For example, Vayanos and Woolley (2013), He and Krishnamurthy (2011), and He and Krishnamurthy (2013) explicitly analyze the delegation relationship between household investors and fund managers and its impacts on asset prices. While such an analysis is interesting and fruitful, it nonetheless neglects some important features of institutional investors such as their risk management constraints. What happens if household investors delegate their financial management to constrained fund managers? How do the delegation relationship and constrained behaviors affect the asset prices? Is financial contagion related to the asset pricing implications generated by delegation? This paper attempts to answer these questions by building an equilibrium delegated asset pricing model with risk management constraints.

In reality, financial institutions are normally regulated entities and are subject to various prudential regulations regarding the

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investment of their own assets. According to the Basel Accords[1], banks and large investment firms are required to follow the value-at-risk (VaR) rules to keep a sufficient amount of capital. Although the VaR rule is widely adopted in the financial industry, its effects on asset prices is a hot topic in academia. For example, Danielsson et al. (2004) argue that VaR leads to an unstable economy that is characterized by low liquidity and high volatility, and Liu et al. (2013) show that its negative impact could be mitigated if fund managers have relative performance concerns. Those papers, however, ignore the fact that institutional investors trade on behalf of household investors and thus the VaR constraint can also affect household investors via delegation.

In this paper, we fill the gap in the literature by analyzing the impact of the VaR constraint on the delegation relationship between fund managers and household investors. In our model, a risk-averse household investor delegates her wealth to a VaR-constrained fund manager, who can also invest her own wealth in the fund. After the delegation, the fund manager invests all the wealth in the capital market. We first solve a baseline model without any constraints and show that all the agents obtain optimal risk sharing. When all agents optimally share the risk, the agent with high risk capacity should absorb a greater share of the risky asset supply. Here, under CARA and normality assumptions, the risk exposure for each agent should be proportional to her risk capacity. In particular, the first optimal risk sharing happens at the trading stage and the second happens at the delegation stage. Thus, the optimal allocations and delegation are all determined by the risk aversion levels of the agents in the economy.

In the setting with the VaR constraint, optimal risk sharing is somehow impaired both at the delegation stage and at the trading stage. Because the constrained manager trades less aggressively, the household investor has to delegate more wealth to the fund manager to achieve a better risk-adjusted return. For this reason, the household investor effectively takes excessive risk due to the existence of the VaR rule. Under the VaR requirement, the risk held by the household investor indeed decreases for each unit of asset. However, because the household investor delegates more wealth to the fund manager, they effectively hold more asset, and thus face more risk. However, in a very simple setting of delegation with VaR requirement, the optimal risk sharing is impaired and household investor takes excessive risk.

Moreover, the presence of VaR also causes the prices of fundamentally uncorrelated assets to become correlated, which leads to a ”contagion effect.” In the unconstrained setting, the prices are linear functions of the supply of various assets, which are independent. For this reason, the prices are independent as well. In the constrained economy, however, the manager faces a volatility constraint and higher volatility in the prices of some assets may cause the constraint to bind, which effectively affects other assets. Thus, asset prices are no longer independent.

Our paper falls into the literature on delegated portfolio management, which studies the behavior of institutional investors and how they affect asset prices. In this strand of literature, the mechanism of delegation between household investors and fund managers is the key to the analysis. Some papers ignore this delegation relationship and use a reduced form compensation to model the features of fund managers, e.g. Basak and Pavlova (2013), Cuoco and Kaniel (2011), Kaniel and Kondor (2013) and Basak et al. (2006). Other papers focus on optimal contracts of delegation, e.g. Bhattacharya and Pfleiderer (1985), Stoughton (1993), Carpenter (2000), Li and Tiwari (2009), Ouyang (2003) and Dybvig et al. (2010), but those papers focus only on either optimal delegation or asset prices, so the analysis is somewhat isolated.

Several papers consider both the delegation and asset pricing implications together, e.g. Vayanos and Woolley (2013), He and Krishnamurthy (2011), He and Krishnamurthy (2013), Buffa and Woolley (2013), Kyle et al. (2011), Breugem and Buss (2019) and Huang (2015). Among those papers, Vayanos and Woolley (2013) do not consider any optimal contracting, and He and Krishnamurthy (2011), He and Krishnamurthy (2013) consider optimal contracting and asset pricing separately. Buffa and Woolley (2013), Kyle et al. (2011) and Huang (2015) jointly determine the optimal contract and asset prices. Those papers, unlike ours, do not consider the constrained behavior of fund managers.

The constraint considered in our paper is VaR-based risk management, which is prevalent in the financial industry. Several papers show that the economy could be unstable when individual market participants are subject to VaR constraints, e.g., Brunnermeier and Pedersen (2007) and Adrian and Shin (2013). In those papers, no delegation is considered. Some papers analyze fund managers' optimal portfolio choice and asset pricing implications. For example, Basak (1995) studies the asset pricing implications of portfolio insurance, and Basak and Shapiro (2001) study that of VaR constraints. Although the agents in their papers are interpreted as fund managers, the authors do not explicitly model the delegation relationship. Fabretti et al. (2014) and Pinar (2013) study portfolio choice policies with delegated portfolio management, and Huang et al. (2019) consider the VaR constraints for fund managers. These papers, however, focus only on the trading stage. Our paper is unique in considering the impact of risk management constraints on both the delegation and trading stages.

One of our key results from the asset pricing model is contagion. The literature on contagion is vast. One type of contagion is across countries (e.g. Eichengreen et al. (1996) and Pesenti and Tille (2005)), and the other type is across asset class (e.g. Hasler and Ornthanalai (2018)). Since contagion generates a higher correlation among assets than that can be explained by the assets' fundamental values, the reason for contagion is extensively analyzed. The explanations of contagion are varied. For example, Hasler and Ornthanalai (2018) consider fluctuating attention and Deng et al. (2018) rely on institutional herding. Moreover, Yuan (2005) combines borrowing constraints and asymmetric information and shows that constrained behaviors lead fundamentally unrelated assets to have correlated prices in equilibrium. Our paper considers comovements across asset classes and combines the VaR constraint and delegation together to generate contagion. Unlike Yuan (2005), our paper does not rely on asymmetric information.

[1] See the published documents: Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version at https://www.bis.org/publ/bcbs128.htm and Basel III: Finalising Post-crisis Reforms at https://www.bis.org/bcbs/publ/d424.htm for details.

The remainder of the paper is organized as follows. Section 2 describes the model settings. Section 3 presents a baseline model without any constraints, and Section 4 solves the model with VaR constraint. Section 5 offers concluding remarks. The Appendix presents all the proofs.

2. The Model

We present a delegated portfolio management model in which VaR-constrained financial institutions (or fund managers) trade on behalf of household investors.

2.1. Assets

We consider a one-period model of trading with $t = 0, 1$: investment starts at time 0 and ends at time 1. There are N risky stocks and one risk-free bond in the market. The bond has an unlimited supply and pays a constant return r. For simplicity, we normalize $r = 0$. Risky stocks pay dividends at $t = 1$, and the payoff of risky stock i is normally distributed: $d_i \sim N(\mu_i, \sigma_i^2)$. For any risky stocks i and j, their payoffs are correlated with covariance σ_{ij}. Throughout the paper, we define the $N \times 1$ vector $\widetilde{\mathbf{D}} = (d_1, d_2, \cdots, d_N)^T$ as the risky assets' payoffs, the $N \times 1$ vector $\boldsymbol{\mu}$ as its mean vector and the $N \times N$ matrix $\boldsymbol{\Sigma}$ as its variance-covariance matrix. The risky stock i has a stochastic supply $\widetilde{s}_i$. We define the $N \times 1$ vector $\widetilde{\mathbf{S}}$ as the stochastic supply. The stochastic supply follows a log-normal distribution: $\ln\widetilde{s}_i \sim N(0, \sigma_{si}^2)$.[2]

For any risky stocks i and j, the covariance between $\ln\widetilde{s}_i$ and $\ln\widetilde{s}_j$ is σ_{sij}. Let the $N \times N$ matrix $\boldsymbol{\Omega}$ be the corresponding variance-covariance matrix. Moreover, we assume that $\widetilde{\mathbf{D}}$ and $\widetilde{\mathbf{S}}$ are independent. At $t = 0$, stock i is traded in the capital market at the price p_i that is endogenously determined in the model. We denote the $N \times 1$ vector $\mathbf{P}$ as the price vector $(p_1, p_2, \cdots, p_N)^T$ for all stocks.

2.2. Agents and Delegation

There are three representative agents in the market: a household investor, a fund manager and a market maker. All agents have utility functions with constant absolute risk aversion (CARA):

$$U_{j1} = -\exp\left[-\tau_j W_{j1}\right], \tag{1}$$

where $j = i, m, d$ denotes the household investor, the fund manager and the market maker, respectively. $\tau_j > 0$ is agent j's coefficient of absolute risk aversion and W_{j1} is the payoff received by agent j at $t = 1$.

The household investor has no access to the market and can only delegate her investment decision to the fund manager before the trading stage. Assume that the household investor has an initial wealth of W_{i0}, and she allocates her initial wealth between a fund and the risk-free bond to maximize her expected utility. We denote by $W \left(0 < \frac{W}{W_{i0}} \leq 1\right)$ the amount of wealth that the household investor allocates to the fund manager. The fund manager has an initial wealth of W_{m0} and invests all her wealth in the fund.[3] The fund manager can trade in the capital market and thus invests the total fund wealth among the risky stocks and the risk-free bond to maximize her expected utility. For the service of delegation, the manager receives a fixed management fee, c, from the household investor.

Based on the initial investment, the household investor holds a proportional share $\frac{W}{W+W_{m0}}$ of the total fund and the manager holds a proportional share $\frac{W_{m0}}{W+W_{m0}}$.[4] Then at $t = 1$, the manager and the household investor share the trading gains of the fund. Denote the $N \times 1$ vector $\mathbf{X} = (x_1, x_2, \cdots, x_N)^T$ as the choices of stocks in the fund portfolio. The household investor's wealth is:

$$W_{i1} = W_{i0} - c + \frac{W}{W + W_{m0}}\mathbf{X}^T\left(\widetilde{\mathbf{D}} - \mathbf{P}\right), \tag{2}$$

and the fund manager's wealth is:

$$W_{m1} = \underbrace{W_{m0} + c}_{\text{Fixed fee}} + \underbrace{\frac{W_{m0}}{W + W_{m0}}\mathbf{X}^T\left(\widetilde{\mathbf{D}} - \mathbf{P}\right)}_{\text{Proportional fee}}. \tag{3}$$

[2] The assumption of a log-normal distribution can simplify our numerical calculation for price correlations. The intuition behind the model does not depend on the specific distributions of the random shocks.

[3] The alternative setting could be that the manager can choose both her wealth invested in the fund W'_{m0} and the portfolio allocation X'. However, because of the properties of CARA utility, for any optimal choice (W'_{m0} and X'), we can always have another choice (W_{m0} and X) that achieves the same payoff distribution for the manager by scaling the risky positions. Thus, to be parsimonious, we simplify the problem by allowing the manager to only chooses the optimal demand X.

[4] There are some other modelling choices for the delegation relationship. Our model is consistent with Vayanos and Wooley (2013), which generates the optimal risk sharing result for delegation. However, the optimal risk sharing result is very general in the literature, e.g., Stoughton (1993).

From equations (2) and (3), we can see that the investor and the manager each hold an equity share of the fund. The stake of each party equals to the proportion of each party's initial wealth invested. The fund manager invests all her wealth W_{m0} in the fund, and therefore the manager holds $\frac{W_{m0}}{W+W_{m0}}$ and the investor holds $\frac{W}{W+W_{m0}}$. The manager's compensation structure is consistent with the widely used linear compensation structure Admati and Pfleiderer, (1997), in which $W_{m0}+c$ represents the fixed fee and $\frac{W_{m0}}{W+W_{m0}}\mathbf{X}^T(\widetilde{\mathbf{D}}-\mathbf{P})$ represents the proportional fee. After the manager determines her optimal demand for the stocks, the risk-averse market maker clears the market by submitting her optimal demand. Define the $N\times 1$ vector $\mathbf{Y}=(y_1,y_2,\cdots,y_N)^T$ as the market maker's demand for risky stocks. The equilibrium stock prices at $t=0$ are determined by the market-clearing condition:

$$\mathbf{X}+\mathbf{Y}=\widetilde{\mathbf{S}}. \tag{4}$$

2.3. The VaR Constraint

In general, institutional investors need to follow risk management practices either to avoid turbulence in the financial market or simply to respond to financial regulations. As discussed in the Introduction, the Basel Accords propose a VaR-based approach to measure market risk, which is widely used in the industry.[5] In our paper, we introduce the VaR constraint into the model:

$$\Pr[E(W_{m1}+W_{i1})-(W_{m1}+W_{i1})\geq V]\leq\widehat{p}, \tag{5}$$

where $W_{m1}+W_{i1}$ is the total fund wealth at $t=1$ and V is a constant, indicating the maximum amount of loss the fund manager is allowed to bear in her portfolio. $\widehat{p}$ is the probability with which the fund manager allows the portfolio wealth loss $E(W_{m1}+W_{i1})-(W_{m1}+W_{i1})$ to be above the constant level V. Thus, the VaR approach controls the probability that a loss goes beyond some fixed level. Following Danielsson, Shin, and Zigrand (2004), it can be shown that the VaR constraint is equivalent to:

$$var(W_{m1}+W_{i1})\leq\left[\frac{V}{\Phi^{-1}(1-\widehat{p})}\right]^2\equiv\widehat{\sigma}^2, \tag{6}$$

where $var(W_{m1}+W_{i1})$ is the variance of fund wealth at $t=1$ and $\Phi(.)$ is the cumulative distribution function of the standard normal distribution. $var(W_{m1}+W_{i1})$ is equivalent to $var(\mathbf{X}^T(\widetilde{\mathbf{D}}-\mathbf{P}))$. Note that both V and $\widehat{p}$ are constants so that we can denote $\left[\frac{V}{\Phi^{-1}(1-\widehat{p})}\right]^2$ as the certain constant $\widehat{\sigma}^2$. For this reason, the original VaR constraint becomes

$$\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}\leq\widehat{\sigma}^2. \tag{7}$$

Therefore, the fund manager needs to keep the volatility of her portfolio below certain threshold.

3. The Baseline Case

We first solve a baseline model in which the fund manager does not face the VaR constraint. In this case, the manager simply maximizes her expected utility. Her optimization problem is defined as follows:

$$\max_{\mathbf{X}}-\exp[-\tau_m W_{m1}],$$

$$\text{s.t.}W_{m1}=W_{m0}+c+\frac{W_{m0}}{W+W_{m0}}\mathbf{X}^T\left(\widetilde{\mathbf{D}}-\mathbf{P}\right). \tag{8}$$

Together with the assumption of normality, we obtain the solution for the familiar mean-variance problem.[6]

$$\mathbf{X}_B=\frac{W+W_{m0}}{\tau_m W_{m0}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{P}), \tag{9}$$

where $\frac{W+W_{m0}}{\tau_m W_{m0}}$ is the effective risk tolerance of the manager, and the subscript B here and in the equations that follow indicates the baseline case. The household investor's problem is to choose an optimal allocation (W) to the fund by maximizing her expected utility, which is

$$\max_{W\geq 0}W_{i0}-c+\frac{W}{W+W_{m0}}\mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P})-\frac{\tau_i}{2}\left(\frac{W}{W+W_{m0}}\right)^2\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}, \tag{10}$$

[5] See the published documents: Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework - Comprehensive Version at https://www.bis.org/publ/bcbs128.htm and Basel III: Finalising Post-crisis Reforms at https://www.bis.org/bcbs/publ/d424.htm for details.

[6] For example, Grossman and Stiglitz (1980).

where W_{i0} and W_{m0} denote the initial wealth of the household investor and of the fund manager respectively. Solving the problem, we obtain:

$$W_B = \frac{\mathbf{X}^T(\boldsymbol{\mu} - \mathbf{P})}{\tau_i \mathbf{X}^T \boldsymbol{\Sigma} \mathbf{X} - \mathbf{X}^T(\boldsymbol{\mu} - \mathbf{P})} W_{m0}, \tag{11}$$

where W_B denotes the optimal wealth allocation of the household investor in the baseline case.

Given the assumptions of normality and CARA utility, the optimal demand of the market maker is $\mathbf{Y} = \frac{1}{\tau_d}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{P})$.[7] Together with the market clearing condition, we can derive the equilibrium prices. Plugging the the equilibrium prices into (9) and (11), we can derive the fund manager's demand and the household investor's allocation in equilibrium. The following proposition gives the results.

Proposition 1. *There exists a unique equilibrium where the stock demand of the fund is*

$$\mathbf{X}_B = \frac{\frac{1}{\tau_m} + \frac{1}{\tau_i}}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \widetilde{\mathbf{S}}, \tag{12}$$

the stock prices are

$$\mathbf{P}_B = \boldsymbol{\mu} - \frac{1}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \boldsymbol{\Sigma} \widetilde{\mathbf{S}}, \tag{13}$$

and the wealth ratio between the household investor and the manager in the benchmark is

$$\frac{W_B}{W_{m0}} = \frac{\tau_m}{\tau_i}. \tag{14}$$

(12) is the total demand of the fund without the VaR constraint. We can see that the fund manager and the household investor together share the total risk in the market with the market maker. Specifically, the agents' risk exposure should be proportional to their risk capacity. The wealth ratio between the household investor and the manager, (14), is the ratio of their risk aversions. Thus, simple delegation achieves the optimal risk-sharing in the baseline model. Given the results of optimal risk sharing, we have the familiar form of the equilibrium price vector (13). Thus, in the baseline case, we have two optimal risk sharing results: the first one is at the trading stage and the second one is at the delegation stage.

4. The Model with the VaR Constraint

In this section, we first solve the equilibrium with the VaR constraint and then examine the impact of such a constraint by comparing the solutions to the baseline case.

4.1. The Equilibrium

In this section, we consider the model with the VaR constraint. In Section 2.3, we wrote the VaR constraint as a constraint on the portfolio variance, thus, we can formalize the manager's optimization problem as follows:

$$\max_{\mathbf{X}} - \exp[-\tau_m W_{m1}],$$

$$\text{s.t.} \quad W_{m1} = W_{m0} + c + \frac{W_{m0}}{W + W_{m0}} \mathbf{X}^T \left(\widetilde{\mathbf{D}} - \mathbf{P} \right),$$

$$\text{and} \quad \mathbf{X}^T \boldsymbol{\Sigma} \mathbf{X} \le \widehat{\sigma}^2. \tag{15}$$

The solution to the above problem depends on whether the constraint is binding. When $\widehat{\sigma}^2$ is very large and the constraint is not binding, the problem is the same as the baseline case. However, when $\widehat{\sigma}^2$ is nontrivial and the constraint is binding, we have different solutions. Moreover, the household investor and the market maker solve the same optimization problem as in the baseline model. The following proposition describes the equilibrium for the model when the VaR constraint is binding.

Proposition 2. *When the fund manager is subject to the VaR constraint, there exists a unique equilibrium if* $\widehat{\sigma} > \widehat{\sigma}^{**} \equiv \frac{\frac{1}{\tau_i}}{\frac{1}{\tau_i} + \frac{1}{\tau_d}} \sqrt{\widetilde{\mathbf{S}}^T \boldsymbol{\Sigma} \widetilde{\mathbf{S}}}$. *Let* $\widehat{\sigma}^* \equiv \frac{\frac{1}{\tau_m} + \frac{1}{\tau_i}}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \sqrt{\widetilde{\mathbf{S}}^T \boldsymbol{\Sigma} \widetilde{\mathbf{S}}}$. *The equilibrium is characterized by the following: I. if* $\widehat{\sigma} \ge \widehat{\sigma}^*$, *the optimal demand of the fund, the price of stocks and the*

[7] It has been proven in the literature that, under CARA utility and normal random shocks, the investor's risky asset demand takes the form of the expected payoff over the payoff variance scaled by the risk aversion parameter. The asset allocations do not depend on initial wealth. See for example Grossman and Stiglitz (1980).

amount of the investor's fund delegation are given by (12),(13) and (14); II. if $\widehat{\sigma}^{**} < \widehat{\sigma} < \widehat{\sigma}^{*}$, the optimal demand of the fund is

$$X = \frac{\widehat{\sigma}}{\sqrt{\mathbf{C}_1^T \boldsymbol{\Sigma}^{-1} \mathbf{C}_1}} \Sigma^{-1} \mathbf{C}_1, \tag{16}$$

the price for stock i is

$$p_i = \mu_i - \tau_d \left(\sum_{j=1}^{N} \sigma_{ij} \widetilde{s}_j - \frac{\widehat{\sigma} \sum_{j=1}^{N} \sigma_{ij} \widetilde{s}_j}{\sqrt{\mathbf{C}_1^T \boldsymbol{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j} \right), \tag{17}$$

and the investor's fund delegation is

$$W = \frac{\tau_d \left(\sqrt{\mathbf{C}_1^T \boldsymbol{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j - \widehat{\sigma} \right)}{\widehat{\sigma}(\tau_i + \tau_d) - \tau_d \sqrt{\mathbf{C}_1^T \boldsymbol{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j} W_{m0}, \tag{18}$$

where $\mathbf{C}_1 = \left(1, \frac{\sum_{j=1}^{N} \sigma_{2j} \widetilde{s}_j}{\sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j}, \cdots, \frac{\sum_{j=1}^{N} \sigma_{Nj} \widetilde{s}_j}{\sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j} \right)^T$. Proposition 2 presents the solutions to the equilibrium when the VaR constraint is binding. When the constraint is not binding, we have the same results as those in Proposition 1. When the constraint is binding, the manager's behavior is constrained, and the optimal risk sharing results no longer hold.

It is straightforward to obtain the effect of the agents' risk aversion coefficients for the case where the fund managers are constrained. The following corollary summarizes the results.

Corollary 1. *In the equilibrium where the VaR constraint is binding, the demand for the stocks is independent of the agents' risk aversion coefficients, the stock prices are independent of the household investor's and the fund manager's risk aversion coefficients and decrease with the market maker's risk aversion coefficient τ_d, and the household investor's fund delegation is independent of the fund manager's risk aversion coefficient, and decreases with the ratio $\frac{\tau_i}{\tau_d}$.*

When the fund manager is constrained, her risk-taking behavior is determined by the VaR constraint. Therefore the VaR constraint parameter $\widehat{\sigma}$, instead of the fund manager's risk aversion coefficent, affects the equilibrium. The stock prices are set by the market maker whose risk aversion coefficient τ_d in turn influences the prices. Furthermore, the price volatility decreases with the risk aversion coefficient of the market marker. Then it is intuitive that the investor's fund delegation is jointly determined by the investor's and the market maker's risk aversion coefficients. In this setting, the fund delegation is determined by the ratio $\frac{\tau_i}{\tau_d}$. Later, we will see that the total risk born by the household investor is also determined by the ratio $\frac{\tau_i}{\tau_d}$.

4.2. The Impact of the VaR Constraint on the Asset Market

To examine the impacts of the VaR constraint, we compare the two equilibria by looking at the optimal demands and equilibrium prices at the trading stage, and the household investor's allocations at the delegation stage. The following proposition summarizes the results.

Proposition 3. *Assume the VaR constraint is binding, i.e., $\widehat{\sigma}^{**} < \widehat{\sigma} < \widehat{\sigma}^{*}$. For asset i, if we compare the optimal demand x_i, equilibrium p_i and household investor's allocation to the fund W to those in the baseline case (x_{Bi}, p_{Bi}, W_B), the following results are true:*

$$x_i < x_{Bi} \tag{19}$$

$$p_i < p_{Bi} \tag{20}$$

and

$$W > W_B. \tag{21}$$

Proposition 1 shows that, compared to the baseline case, when the VaR constraint is binding, the optimal demand and prices are lower and the household investor's allocation is higher. Intuitively, due to the volatility constraint, the fund manager has to trade less aggressively to keep portfolio volatility below the threshold, which in turn drives the equilibrium prices of risky assets lower. Thus, the risk sharing at the trading stage becomes suboptimal.

At the delegation stage, (21) shows that the household investor actually invests more in the fund than she would do in the baseline case without the VaR constraint. The only way that the household investor can trade risky assets is through delegation, so she needs to adjust her optimal allocation based on the fund manager's behavior. Because the fund manager trades less aggressively, the household investor has to invest more to achieve a better risk-return trade-off. For this reason, risk sharing optimality at the delegation stage is also lost. Define the total risk the household investor bears as $Risk \equiv \left(\frac{W}{W+W_{m0}}\right)\sqrt{\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}}$. Denote as $Risk_B$ the total risk the household investor bears in the benchmark case, i.e., $Risk_B = \left(\frac{W_B}{W_B+W_{m0}}\right)\sqrt{\mathbf{X}_\mathbf{B}^T\boldsymbol{\Sigma}\mathbf{X}_\mathbf{B}}$. The following corollary illustrates the impact of the VaR constraint on $Risk$.

Corollary 2. *When the VaR constraint is binding, the household investor effectively takes on more risk than she would do in the baseline case; that is*

$$Risk > Risk_B,$$

and the household investor's total risk decreases with the ratio $\frac{\tau_i}{\tau_d}$. Without optimal risk sharing at the delegation stage, the household investor effectively bears more risk. Under the VaR requirement, the risk held by the household investor indeed decreases for each unit of asset. However, because the household investor delegates more wealth to the fund manager, they effectively hold more asset, and thus face more risk.

4.3. A Simplified Two-Asset Model

In this section, we construct a simplified two-assets model to illustrate the results from the multiasset model above more clearly. Assume that there are only two independent stocks in the model. The supply of each stock is $\tilde{s}_1$ and $\tilde{s}_2$, which are log-normally distributed: $\ln\tilde{s}_i \sim N(0, \sigma_{si}^2)$, i = 1, 2. The payoff of the stocks is normally distributed: $d_i \sim N(\mu_i, \sigma_i^2)$, i = 1, 2. Both the supply and payoff of the two stocks are independent. The VaR constraint becomes $x_1^2\sigma_1^2 + x_2^2\sigma_2^2 \le \hat{\sigma}^2$. With Proposition 1 and Proposition 2, we can obtain the optimal stock demand of the fund manager, the equilibrium prices and the investor's optimal fund delegation for the baseline model and the constrained model.

We illustrate our results with numerical examples. Let $\tau_m = \tau_d = 1, \tau_i = 2, \sigma_1 = \sigma_2 = 1, \mu_1 = \mu_2 = 1, W_{m0} = 1, \tilde{s}_1 = \tilde{s}_2 = 1$. Then the boundaries $\hat{\sigma}^*$ and $\hat{\sigma}^{**}$ are given by $\frac{3\sqrt{2}}{5}$ and $\frac{\sqrt{2}}{3}$. For the baseline model, we have

$$x_{A1} = x_{A2} = 0.6, p_{A1} = p_{A2} = 0.6, W_B = 0.5, Risk_B = \frac{\sqrt{2}}{5}$$

For the constrained model, when $\hat{\sigma} \ge \hat{\sigma}^*$, the VaR constraint is not binding and the solution is identical to that of the baseline case. However, when $\hat{\sigma}^{**} < \hat{\sigma} < \hat{\sigma}^*$, we obtain the following binding solution:

$$x_1 = x_2 = \frac{\hat{\sigma}}{\sqrt{2}}, p_1 = p_2 = \frac{\hat{\sigma}}{\sqrt{2}}, W = \frac{\sqrt{2} - \hat{\sigma}}{3\hat{\sigma} - \sqrt{2}}, Risk = \frac{\sqrt{2} - \hat{\sigma}}{2}$$

Panels (a) and (b) of Figure 1 show that both demand and prices are lower for both stocks when the VaR constraint is binding; compared with the baseline model, we find that both of them increase with $\hat{\sigma}$. Panels (c) and (d) of Figure 1 show that the investor's allocation of wealth to the fund and the total risk are both higher when the VaR constraint is binding, and they decrease with $\hat{\sigma}$.

In Figure 2, we demonstrate how changes in agents' risk aversion influence the results. From Corollary 1, we know that the fund manager's risk aversion does not play a role in the equilibrium when the constraint is binding, so we fix the fund manager's risk aversion to be $\tau_m = 1$ and vary the household investor's and the market maker's risk aversion. We consider three cases: i) $\tau_i = 1$ and $\tau_d = 1$; ii) $\tau_i = 2$ and $\tau_d = 1$; iii) $\tau_i = 2$ and $\tau_d = 2$. All other parameters remain the same as before. Consistent with Corollary 1, when we focus on the region where the VaR constraint is binding, Panel (a) shows that neither the investor's nor the market maker's risk aversion affects the optimal demand. Panel (b) shows that stock prices are not affected by the household investor's risk aversion and decrease with the market maker's risk aversion. Panels (c) and (d) show that both the wealth allocation of the household investor and the total risk taken decrease with the ratio $\frac{\tau_i}{\tau_d}$ when the constraint is binding. More importantly, the results in Proposition 1 and Corollary 2 still hold with different risk aversion coefficients.

4.4. Stock Price Contagion

We have shown in the previous sections that since the VaR constraint impairs optimal risk sharing at the trading stage, it affects the relationship among stock prices. To simplify the idea, we assume that the stochastic supplies of all risky stocks are independent. Given that the equilibrium prices solved for in (13) are linear functions of the stock supply, the stock prices are also independent, so their covariance is 0 in the baseline case. When the VaR constraint is binding, the stock prices are no longer independent.

Corollary 3. *Given that the fundamental values and the stochastic supply are independent across stocks and that the VaR constraint is binding, the correlation between any two stock prices is*

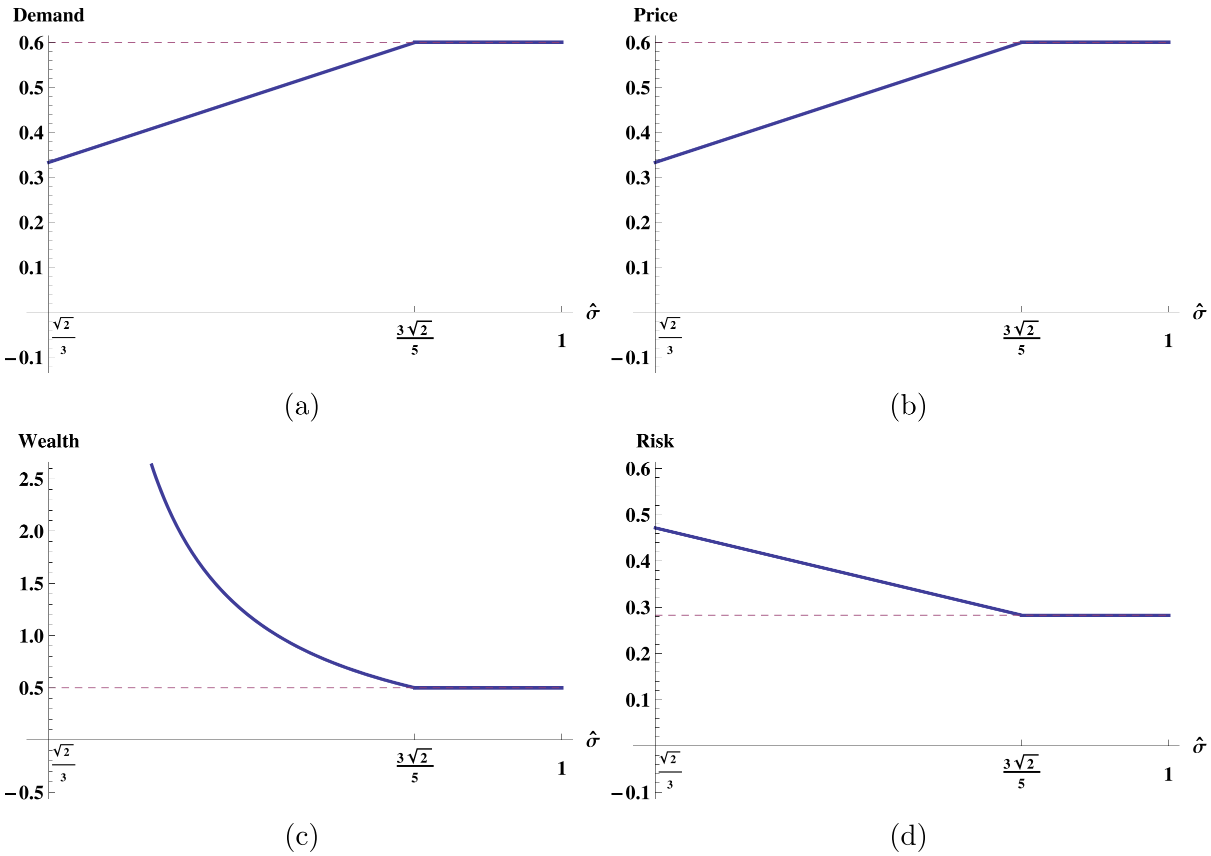

Fig. 1. The Equilibrium of Two Asset Markets Figure 1 plots the equilibrium solution of the two cases against the VaR constraint parameter $\hat{\sigma}$. The dashed line shows the benchmark case and the solid line shows the constrained case. There are two stocks $i = 1,2$. For parameter values, let $\tau_m = \tau_d = 1, \tau_i = 2, \sigma_1 = \sigma_2 = 1, \mu_1 = \mu_2 = 1, W_{m0} = 1, \widetilde{s}_1 = \widetilde{s}_2 = 1$. $\widehat{\sigma}^*$ and $\widehat{\sigma}^{**}$ are given by $\frac{3\sqrt{2}}{5}$ and $\frac{\sqrt{2}}{3}$, respectively. The solutions of the demand and the price are identical for asset 1 and 2 with the parameter values.

$$Corr(p_i, p_j) = Corr\left[\widetilde{s}_i\left(1 - \frac{\widehat{\sigma}}{\sqrt{\sum_{k=1}^{N}\sigma_k^2\widetilde{s}_k^2}}\right), \widetilde{s}_j\left(1 - \frac{\widehat{\sigma}}{\sqrt{\sum_{k=1}^{N}\sigma_k^2\widetilde{s}_k^2}}\right)\right]. \tag{22}$$

Thus, the constrained behavior of the fund managers leads otherwise independent assets to be correlated, and we call this effect the “contagion effect”.[8] In the unconstrained economy, the prices are linear to the supply of assets, which is independent across assets. For this reason, the prices are independent as well. In the constrained economy, however, the more volatile prices of some assets may render the volatility constraint binding, which effectively affects other asset prices. Thus, the correlation between asset prices is no longer zero.

To illustrate our idea, we calculate the average stock price correlation by Monte Carlo simulation. For each $\widehat{\sigma}$, we first compute the equilibrium prices using 10000 simulated paths of $\widetilde{s}_1$ and $\widetilde{s}_2$, assuming $ln(\widetilde{s}_i) \sim N(0, 0.5), i = 1,2$. We keep the price points where an equilibrium exists, i.e. $\widehat{\sigma} > \widehat{\sigma}^{**}$, and calculate the correlation between the two stock prices. Then we repeat this exercise 20 times and compute the average correlation. Figure 3 illustrates the change in stock price correlation as we increase the strength of the VaR constraint.

In the simulation, we set the lower bound of $\widehat{\sigma}$ to be 0.5 so as to ensure that the equilibrium exists, i.e., $\widehat{\sigma}^{**} \leq \widehat{\sigma}$, with high probability. As the VaR constraint parameter $\widehat{\sigma}$ increases from its lower end, the constraint goes from extremely binding to less binding and then to nonbinding, that is, the constraints loosen as we increase $\widehat{\sigma}$. Correspondingly, the stock price correlation first increases, then peaks and finally decreases to zero. There are two effects working in opposite directions. First, the VaR constraint generates

[8] A similar definition is adopted by Yuan (2005).

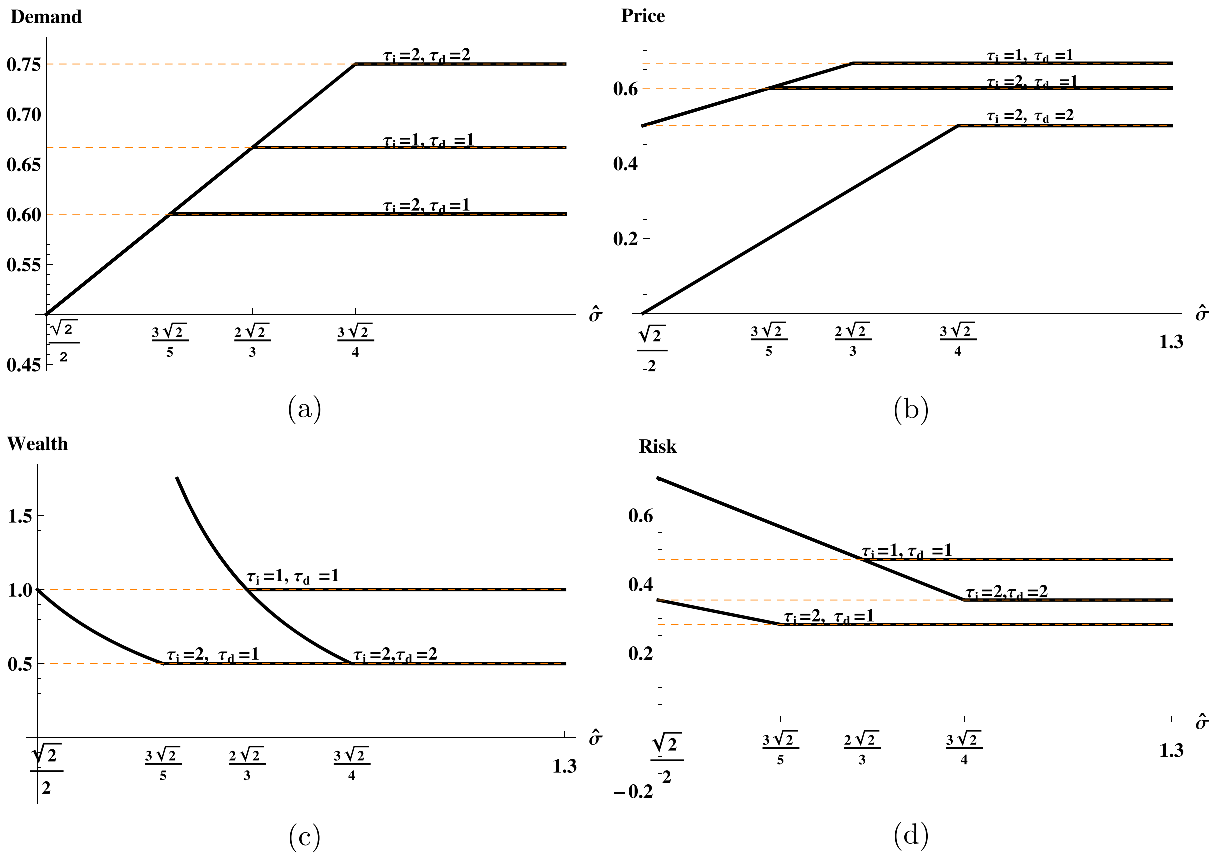

Fig. 2. Figure 2 The Effect of Agents' Risk Aversion Figure 2 plots the equilibrium of the two-asset cases against the VaR constraint parameter $\widehat{\sigma}$ for different values of risk aversion coefficients. The dashed line shows the benchmark case and the solid line shows the case with the VaR constraint. There are two stocks $i = 1, 2$. Unless specified in the plots, the parameter values are: $\tau_m = 1$ $\sigma_1 = \sigma_2 = 1, \mu_1 = \mu_2 = 1$, $W_{m0} = 1, \widetilde{s}_1 = \widetilde{s}_2 = 1$.

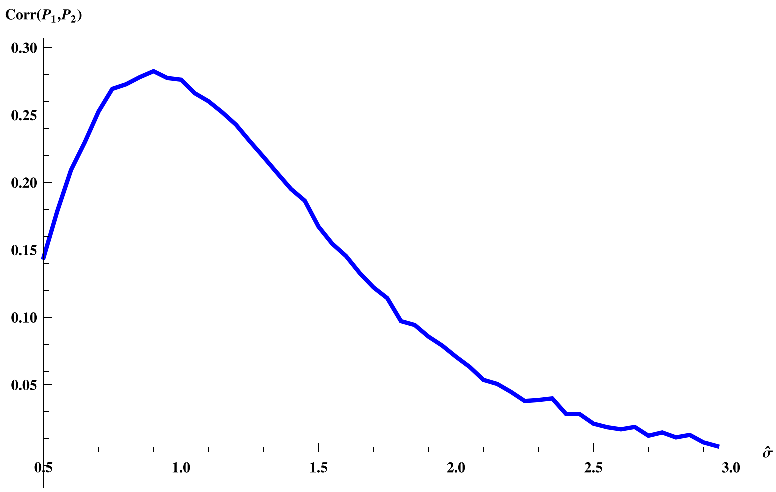

Fig. 3. Figure 3 Stock Price Correlations Figure 3 plots the correlation between the two stock prices against the VaR constraint parameter $\widehat{\sigma}$. There are two stocks $i = 1, 2$. *For each* $\widehat{\sigma}$, *the stock prices are simulated with 10000 paths (realizations of random shocks) for 20 times, and the plot shows the average correlation coefficient between the two stock prices. Other parameters are* $\tau_m = \tau_d = 1, \tau_i = 2$, $\sigma_1 = \sigma_2 = 1$, $\mu_1 = \mu_2 = 1$, $W_{m0} = 1$, and $\ln \widetilde{s}_i \sim N(0, 0.5)$, $i = 1, 2$.

comovement in the fund manager's demands for the two stocks. Therefore, as the VaR constraint becomes stronger, the comovement effect increases the correlation between stock prices. Second, when the VaR constraint is stronger, the fund manager trades less aggressively in the stock market, which weakens the fund manager's impact on stock prices. In the extreme case where $\hat{\sigma}$ is close to zero, the manager does not trade in the stock market and stock prices are also uncorrelated. We can see from Figure 3 that the first effect dominates when VaR constraint is binding at a low to intermediate level; the second effect dominates when the constraint is very tight.

Yuan (2005) combines borrowing constraints and asymmetric information and shows that constrained behaviors lead fundamentally unrelated assets to have correlated prices in equilibrium. Our definition of contagion is the same as in Yuan (2005). Although contagion is also driven by financial constraints, our channel of contagion is different from that of Yuan (2005). First, in Yuan (2005), contagion happens because uninformed traders are confused about borrowing constraints. Our results, however, do not rely on asymmetric information. Second, our paper explicitly considers the delegation relationship between fund managers and household investors, which is not in Yuan (2005).

5. Conclusion

Our paper presents a multi-asset pricing model of delegation in which fund managers are subject to a VaR constraint. We find that the existence of such VaR risk regulation distorts the risk-sharing mechanism and causes price contagion. Once the fund manager is bound by the VaR constraint, she holds fewer risky assets in equilibrium, thus distorting the optimal risk sharing in trading. The household investor has to allocate more wealth to the fund to achieve a better risk-return trade-off, and thus bears more risk. For this reason, the optimal risk sharing in the delegation stage is also distorted. In contrast to the baseline model, the stock prices of fundamentally uncorrelated assets become positively correlated when the VaR constraint is in play. However, the relationship between the stock price correlation and the tightness of the constraint is not monotonic because as the constraint becomes more stringent, the fund manager will hold fewer assets, and her impact on asset prices will therefore be reduced.

Appendix A

A1. Proof of Proposition 2 and Corollary 1

We only show the proof of Proposition 2 because Proposition 1 is just the unconstrained case of Proposition 2. The household investor takes $\mathbf{X}$ and $\mathbf{P}$ as given. With normal random shocks and CARA utility, she solves the following problem:

$$\max_{W} W_{i0} - c + \frac{W}{W+W_{m0}}\mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P}) - \frac{\tau_i}{2}\left(\frac{W}{W+W_{m0}}\right)^2\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}, \tag{23}$$

where W_{i0} and W_{m0} denote the initial wealth of the household investor and of the fund manager respectively. It is equivalent to choose an optimal value for $\frac{W}{W+W_{m0}}$, which can be solved from the above quadratic as,

$$\frac{W}{W+W_{m0}} = \frac{\mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P})}{\tau_i\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}}, \tag{24}$$

from which we have

$$\frac{W}{W_{m0}} = \frac{\mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P})}{\tau_i\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X} - \mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P})}. \tag{25}$$

The fund manager takes W and $\mathbf{P}$ as given, and solves the following problem:

$$\max_{\mathbf{X}} W_{m0} + c + \frac{W_{m0}}{W+W_{m0}}\mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P}) - \frac{\tau_m}{2}\left(\frac{W_{m0}}{W+W_{m0}}\right)^2\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}, \tag{26}$$

$$\text{s.t. } \mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X} \leq \hat{\sigma}^2.$$

We set up the Lagrangian:

$$\mathbf{L} = W_{m0} + c + \frac{W_{m0}}{W+W_{m0}}\mathbf{X}^T(\boldsymbol{\mu}-\mathbf{P}) - \frac{\tau_m}{2}\left(\frac{W_{m0}}{W+W_{m0}}\right)^2\mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X} + \lambda\left(\hat{\sigma}^2 - \mathbf{X}^T\boldsymbol{\Sigma}\mathbf{X}\right). \tag{27}$$

By Kuhn-Tucker conditions, we have

$$\frac{\partial\mathbf{L}}{\partial\mathbf{X}} = \frac{W_{m0}(\boldsymbol{\mu}-\mathbf{P})}{W+W_{m0}} - \tau_m\left(\frac{W_{m0}}{W+W_{m0}}\right)^2\boldsymbol{\Sigma}\mathbf{X} - 2\lambda\boldsymbol{\Sigma}\mathbf{X} = 0, \tag{28}$$

$$\frac{\partial \mathbf{L}}{\partial \lambda} = \hat{\sigma}^2 - \mathbf{X}^T \mathbf{\Sigma} \mathbf{X} \geq 0, \quad \lambda \geq 0, \quad \lambda \frac{\partial \mathbf{L}}{\partial \lambda} = 0. \tag{29}$$

Case I. $\lambda = 0$ and $\mathbf{X}^T \mathbf{\Sigma} \mathbf{X} < \hat{\sigma}^2$: The above Kuhn-Tucker condition yields,

$$\mathbf{X} = \frac{(W + W_{m0})\Sigma^{-1}(\boldsymbol{\mu} - \mathbf{P})}{\tau_m W_{m0}}, \tag{30}$$

Given the optimal demand of the market maker $\mathbf{Y} = \frac{\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{P})}{\tau_d}$, and by market clearing conditions:

$$\mathbf{X} + \mathbf{Y} = \widetilde{\mathbf{S}},$$

we can the equilibrium asset prices can be written as follows,

$$\mathbf{P}_{\text{B}} = \boldsymbol{\mu} - \frac{1}{\frac{1}{\tau_m}\left(\frac{W+W_{m0}}{W_{m0}}\right) + \frac{1}{\tau_d}} \mathbf{\Sigma} \widetilde{\mathbf{S}}. \tag{31}$$

Then we derive $\frac{W}{W_{m0}}$ for this case. By (30), we have

$$\mathbf{X}^T(\boldsymbol{\mu} - \mathbf{P}) = \frac{(\boldsymbol{\mu} - \mathbf{P})^T \Sigma^{-1} (\boldsymbol{\mu} - \mathbf{P})}{\tau_m} \left(\frac{W + W_{m0}}{W_{m0}}\right), \tag{32}$$

and

$$\mathbf{X}^T \Sigma \mathbf{X} = \frac{(\boldsymbol{\mu} - \mathbf{P})^T \Sigma^{-1} (\boldsymbol{\mu} - \mathbf{P})}{\tau_m^2} \left(\frac{W + W_{m0}}{W_{m0}}\right)^2. \tag{33}$$

Substitute the above expressions into equation (25), the expression $(\boldsymbol{\mu} - \mathbf{P})^T \Sigma^{-1} (\boldsymbol{\mu} - \mathbf{P})$ is canceled out. We can then solve $\frac{W}{W_{m0}}$ which is the same as in Proposition 1. The fund manager's demand $\mathbf{X}$ and the price $\mathbf{P}$ is immediate. In addition, the condition $\mathbf{X}^T \mathbf{\Sigma} \mathbf{X} < \hat{\sigma}^2$ is equivalent to $\hat{\sigma} \geq \frac{\frac{1}{\tau_m} + \frac{1}{\tau_i}}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \sqrt{\widetilde{\mathbf{S}}^T \mathbf{\Sigma} \widetilde{\mathbf{S}}}$ by substituting the price equation into the demand function.

Case II. $\lambda > 0$ and $\mathbf{X}^T \mathbf{\Sigma} \mathbf{X} = \hat{\sigma}^2$ (i.e., the VaR constraint is binding and $\hat{\sigma} < \frac{\frac{1}{\tau_m} + \frac{1}{\tau_i}}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \sqrt{\widetilde{\mathbf{S}}^T \mathbf{\Sigma} \widetilde{\mathbf{S}}}$): The Kuhn-Tucker condition yields,

$$\mathbf{X} = \frac{\frac{W_{m0}}{W+W_{m0}}}{\tau_m \left(\frac{W_{m0}}{W+W_{m0}}\right)^2 + 2\lambda} \Sigma^{-1}(\boldsymbol{\mu} - \mathbf{P}), \tag{34}$$

Similar as in Case I, Given the optimal demand of the market maker $\mathbf{Y} = \frac{\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{P})}{\tau_d}$, and by market clearing conditions:

$$\mathbf{X} + \mathbf{Y} = \widetilde{\mathbf{S}},$$

we can the equilibrium asset prices can be written as follows,

$$\mathbf{P} = \boldsymbol{\mu} - \frac{1}{\frac{\frac{W_{m0}}{W+W_{m0}}}{\tau_m \left(\frac{W_{m0}}{W+W_{m0}}\right)^2 + 2\lambda} + \frac{1}{\tau_d}} \mathbf{\Sigma} \widetilde{\mathbf{S}}. \tag{35}$$

Then we solve λ as an expression of W/W_{m0}. We have from the above derivation,

$$\mathbf{X}^T(\boldsymbol{\mu} - \mathbf{P}) = \left(\frac{\frac{W_{m0}}{W+W_{m0}}}{\tau_m \left(\frac{W_{m0}}{W+W_{m0}}\right)^2 + 2\lambda}\right) (\boldsymbol{\mu} - \mathbf{P})^T \Sigma^{-1} (\boldsymbol{\mu} - \mathbf{P}), \tag{36}$$

and

$$\mathbf{X}^T \Sigma \mathbf{X} = \left(\frac{\frac{W_{m0}}{W+W_{m0}}}{\tau_m \left(\frac{W_{m0}}{W+W_{m0}}\right)^2 + 2\lambda}\right)^2 (\boldsymbol{\mu} - \mathbf{P})^T \Sigma^{-1} (\boldsymbol{\mu} - \mathbf{P}). \tag{37}$$

Substitute the above expressions into equation (25), the expression $(\boldsymbol{\mu} - \mathbf{P})^T\Sigma^{-1}(\boldsymbol{\mu} - \mathbf{P})$ is canceled out. We can have the following expression for λ,

$$\frac{\frac{W_{m0}}{W+W_{m0}}}{\tau_m\left(\frac{W_{m0}}{W+W_{m0}}\right)^2 + 2\lambda} = \frac{W + W_{m0}}{\tau_i W}. \tag{38}$$

Substitute the above expression into the equation for $\mathbf{X}$ and $\mathbf{P}$, we have $\mathbf{X} = \frac{W+W_{m0}}{\tau_i W}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{P})$ and $\mathbf{P} = \boldsymbol{\mu} - \frac{1}{\frac{W+W_{m0}}{\tau_i W}+\frac{1}{\tau_d}}\boldsymbol{\Sigma}\widetilde{\mathbf{S}}$. Then we can solve $\frac{W_{m0}}{W+W_{m0}}$ (18) from

$$\mathbf{X}^T\Sigma\mathbf{X} = \left(\frac{\frac{W+W_{m0}}{\tau_i W}}{\frac{W+W_{m0}}{\tau_i W}+\frac{1}{\tau_d}}\right)^2 \widetilde{\mathbf{S}}^T\Sigma\widetilde{\mathbf{S}} = \widehat{\sigma}^2. \tag{39}$$

The expression (16) and (17) can be obtained immediately. Wealth delegation $W \geq 0$ implies $\frac{W+W_{m0}}{W} \geq 1$. Applying $\frac{W+W_{m0}}{W} \geq 1$ to (39), we derive the lower bound of $\widehat{\sigma}$ as $\frac{\frac{1}{\tau_i}}{\frac{1}{\tau_i}+\frac{1}{\tau_d}}\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}$. Finally, Corollary 1 is immediate by observing equations (16), (17), and (18). Q.E.D.

A2. Proof of Proposition 1 and Corollary 2

By Proposition 1 and 2 and $\mathbf{C}_1 = \frac{\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}{\sum_{j=1}^{N}\sigma_{1j}\widetilde{s}_j}$ we have

$$\frac{x_B - x_i}{\widetilde{s}_i} = \frac{\frac{1}{\tau_m}+\frac{1}{\tau_i}}{\frac{1}{\tau_m}+\frac{1}{\tau_i}+\frac{1}{\tau_d}} - \frac{\widehat{\sigma}}{\sqrt{\mathbf{C}_1^T\boldsymbol{\Sigma}^{-1}\mathbf{C}_1}\sum_{j=1}^{N}\sigma_{1j}\widetilde{s}_j},$$

$$= \frac{\frac{1}{\tau_m}+\frac{1}{\tau_i}}{\frac{1}{\tau_m}+\frac{1}{\tau_i}+\frac{1}{\tau_d}} - \frac{\widehat{\sigma}}{\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}},$$

$$p_B - p_i = \tau_d\left(\sum_{j=1}^{N}\sigma_{ij}\widetilde{s}_j - \frac{\widehat{\sigma}\sum_{j=1}^{N}\sigma_{ij}\widetilde{s}_j}{\sqrt{\mathbf{C}_1^T\boldsymbol{\Sigma}^{-1}\mathbf{C}_1}\sum_{j=1}^{N}\sigma_{1j}\widetilde{s}_j}\right) - \frac{\sum_{j=1}^{N}\sigma_{ij}\widetilde{s}_j}{\frac{1}{\tau_m}+\frac{1}{\tau_i}+\frac{1}{\tau_d}}$$

$$= \tau_d\sum_{j=1}^{N}\sigma_{ij}\widetilde{s}_j\left(\frac{\frac{1}{\tau_m}+\frac{1}{\tau_i}}{\frac{1}{\tau_m}+\frac{1}{\tau_i}+\frac{1}{\tau_d}} - \frac{\widehat{\sigma}}{\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}}\right),$$

$$\frac{W_B - W}{W_{m0}} = \frac{\tau_m}{\tau_i} - \frac{\tau_d\left(\sqrt{\mathbf{C}_1^T\boldsymbol{\Sigma}^{-1}\mathbf{C}_1}\sum_{j=1}^{N}\sigma_{1j}\widetilde{s}_j - \widehat{\sigma}\right)}{\widehat{\sigma}(\tau_i+\tau_d) - \tau_d\sqrt{\mathbf{C}_1^T\boldsymbol{\Sigma}^{-1}\mathbf{C}_1}\sum_{j=1}^{N}\sigma_{1j}\widetilde{s}_j}$$

$$= \frac{(\tau_i\tau_d + \tau_d\tau_m + \tau_m\tau_i)\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}}{\tau_i\left[\widehat{\sigma}(\tau_i+\tau_d) - \tau_d\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}\right]}\left(\frac{\widehat{\sigma}}{\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}} - \frac{\frac{1}{\tau_m}+\frac{1}{\tau_i}}{\frac{1}{\tau_m}+\frac{1}{\tau_i}+\frac{1}{\tau_d}}\right).$$

(Note that $\widehat{\sigma}(\tau_i+\tau_d) - \tau_d\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}} > 0$ when the investor's allocation to the fund is positive.) And

$$Risk_B - Risk = \frac{\frac{1}{\tau_i}\sqrt{\widetilde{\mathbf{S}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{S}}}}{\frac{1}{\tau_m}+\frac{1}{\tau_i}+\frac{1}{\tau_d}} - \frac{\tau_d\left(\sqrt{\mathbf{C}_1^T\boldsymbol{\Sigma}^{-1}\mathbf{C}_1}\sum_{j=1}^{N}\sigma_{1j}\widetilde{s}_j - \widehat{\sigma}\right)}{\tau_i}$$

$$= \frac{\tau_d \sqrt{\mathbf{C}_1^T \mathbf{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j}{\tau_i} \left(\frac{\widehat{\sigma}}{\sqrt{\widetilde{\mathbf{S}}^T \mathbf{\Sigma} \widetilde{\mathbf{S}}}} - \frac{\frac{1}{\tau_m} + \frac{1}{\tau_i}}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \right).$$

The results for stock holdings, prices, wealth allocations and the portfolio risk follow immediately from the condition $\widehat{\sigma} < \frac{\frac{1}{\tau_m} + \frac{1}{\tau_i}}{\frac{1}{\tau_m} + \frac{1}{\tau_i} + \frac{1}{\tau_d}} \sqrt{\widetilde{\mathbf{S}}^T \mathbf{\Sigma} \widetilde{\mathbf{S}}}$ when the VaR constraint is binding.

Finally, *Risk* decreases in $\frac{\tau_i}{\tau_d}$ by

$$Risk = \frac{\tau_d \left(\sqrt{\mathbf{C}_1^T \mathbf{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j - \widehat{\sigma} \right)}{\tau_i}$$

Q.E.D.

A3. Proof of Corollary 3

From Proposition 2, we have $p_i = \mu_i - \tau_d \left(\sum_{j=1}^{N} \sigma_{ij} \widetilde{s}_j - \frac{\widehat{\sigma} \sum_{j=1}^{N} \sigma_{ij} \widetilde{s}_j}{\sqrt{\mathbf{C}_1^T \mathbf{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j} \right)$, where $\mathbf{C}_1 = \left(1, \frac{\sum_{j=1}^{N} \sigma_{2j} \widetilde{s}_j}{\sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j}, \cdots, \frac{\sum_{j=1}^{N} \sigma_{Nj} \widetilde{s}_j}{\sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j} \right)^T$.

It can be seen that $\left(\sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j \right) \mathbf{C}_1 = \mathbf{\Sigma} \, \widetilde{\mathbf{S}}$. Therefore, $\sqrt{\mathbf{C}_1^T \mathbf{\Sigma}^{-1} \mathbf{C}_1} \sum_{j=1}^{N} \sigma_{1j} \widetilde{s}_j = \sqrt{\widetilde{\mathbf{S}} \mathbf{\Sigma} \, \widetilde{\mathbf{S}}}$. As the fundamental is independent across stocks, i.e., $\sigma_{ij} = 0$ for $i \neq j$, we have

$\sqrt{\widetilde{\mathbf{S}} \mathbf{\Sigma} \, \widetilde{\mathbf{S}}} = \sqrt{\sum_{k=1}^{N} \sigma_k^2 \widetilde{s}_k^2}$ and $\sum_{j=1}^{N} \sigma_{ij} \widetilde{s}_j = \sigma_i^2 \widetilde{s}_i$. Therefore, $p_i = \mu_i - \tau_d \sigma_i \widetilde{s}_i \left(1 - \frac{\widehat{\sigma}}{\sqrt{\sum_{k=1}^{N} \sigma_k^2 \widetilde{s}_k^2}} \right)$. Then equation (22) is straightforward.

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